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Delayed feedback of sampled higher derivatives

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Systems where the present rate of change of the state depends on the past values of the higher rates of change of the state are described by so-called advanced functional differential equations (AFDEs). In an AFDE, the highest derivative of the state-space coordinate appears with delayed argument only. The corresponding linearized equations are always unstable with infinitely many unstable poles, and are rarely related to practical applications due to their inherently implicit nature. In this paper, one of the simplest AFDEs, a linear scalar first-order system, is considered with the delayed feedback of the second derivative of the state in the presence of sampling in the feedback loop (i.e. in the case of digital control). It is shown that sampling of the feedback may stabilize the originally infinitely unstable system for certain parameter combinations. The result explains the stable behaviour of certain dynamical systems with feedback delay in the highest derivative.

Keywords: functional differential equations; advanced argument; sampling; stability; delay

1. Introduction

Functional differential equations (FDEs) are often used to describe hereditary systems in different fields of science and engineering. FDEs can be categorized into retarded, neutral and advanced types (e.g. Èl'sgol'c 1964; Kolmanovskii & Myshkis 1999). If the rate of change of the state depends on the past states of the system, then the corresponding mathematical model is a retarded functional differential equation (RFDE). If the rate of change of the state depends on its own past values as well, then the corresponding equation is called a neutral functional differential equation (NFDE). If the rate of change of the state depends on the past values of higher derivatives of the state, then the system is described by an advanced functional differential equation (AFDE). Note that these equations are also referred to as FDEs of retarded, neutral or advanced type.

The reason for the word 'advanced' can be demonstrated by the following example. Consider the simple AFDE

$$\dot{x}(t) = \ddot{x}(t - \tau). \tag{1.1}$$

By a τ-shift transformation in time, and by using the new variable $z = \dot{x}$, this equation can be written in the form

$$\dot{z}(t) = z(t + \tau). \tag{1.2}$$

Here, the rate of change of the state is determined by the future values of the state, i.e. an advanced state determines the present rate of change of the state.

As opposed to RFDEs and NFDEs, AFDEs are rarely used in practical applications due to their inverted causality explained by equation (1.2). Furthermore, the solutions of AFDEs, in general, lose their initial smoothness, and the method of successive integration shows that solutions may not even exist, while the two-sided solutions do exist (Wiener 1994). Forward solution of AFDEs corresponds to backward solution of RFDEs, which may not exist either. While linear autonomous RFDEs have infinitely many poles on the left half of the complex plane, linear autonomous AFDEs have infinitely many poles on the right half of the complex plane. In this sense, linear autonomous AFDEs are always strongly or infinitely unstable.

The literature of FDEs is quite extensive; we refer the interested reader for example to the texts by Kolmanovskii & Nosov (1986), Stepan (1989), Hale & Lunel (1993), Niculescu (2001), Hu & Wang (2002) or Michiels & Niculescu (2007), just to mention a few. However, these works mostly deal with RFDEs and NFDEs, and AFDEs are not considered since they are said to have no practical significance. Some basic properties of AFDEs together with some special cases are described in the books of Èl'sgol'c (1964), Wiener (1994) and Kolmanovskii & Myshkis (1999). Oscillatory conditions for some special AFDEs are discussed in the basic paper by Ladas & Stavroulakis (1982). Shah & Wiener (1983) and Song *et al.* (2005) analysed systems with piecewise constant advanced arguments. FDEs with alternately advanced and retarded arguments were analysed by Filho (1997), Rodrigues (1997) and Lv *et al.* (2007).

In this paper, a linear first-order scalar AFDE is considered with piecewise constant argument in the advanced term. The investigation of this system is motivated by an engineering example of a digital control system with the delayed feedback of a derivative higher than that of the present state. The piecewise constant argument corresponds to the sampling effect (zero-order hold, ZOH) in the digital control. It is shown that the system can be exponentially stable, although the continuous counterpart, the autonomous AFDE with continuous argument in the advanced term, is unstable with infinitely many unstable poles.

2. Motivation

The motivation of the current study arose during the analysis of a simple stick balancing problem via feedback control that uses accelerometers to determine the stick's angular position. The mechanical model can be seen in figure 1. The equation of motion reads

$$J\ddot{\varphi} - Hmg\sin\varphi = Q, \tag{2.1}$$

where φ is the angular position of the pinned stick, m the mass, H the distance between the suspension point O and the centre of gravity C, J the mass moment of inertia with respect to the axis normal to the plane of the figure through point
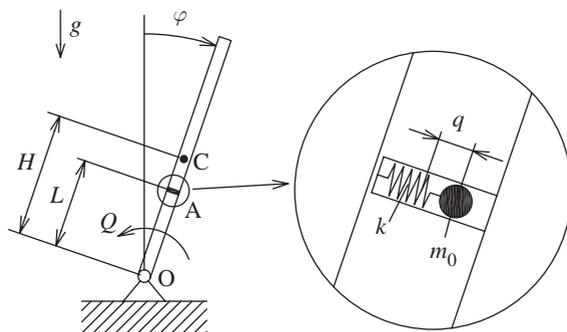
Figure 1. Model of stick balancing with one degree of freedom, and the mechanical model of the accelerometer.

O and Q the control torque. This model, also called stiffness control, is used to describe self-balancing of humans in the sagittal plane (Winter *et al.* 1998; Stepan 2009).

In figure 1, the enlarged insert represents the mechanical model of the piezo-accelerometer attached to the stick at point A. This accelerometer operates normal to the stick as a mass m_0 attached to a spring of stiffness k modelling the piezo crystal. The accelerometer's output y is proportional to the displacement q of the mass: $y = K_A q$ with K_A (V m^{-1}) being the characteristic constant of the accelerometer. The displacement q of the mass depends on the angular position φ of the rod. If the rod is standing still in an oblique position, then

$$q = \frac{m_0 g}{k} \sin \varphi. \tag{2.2}$$

In an ideal linear case, the output is proportional to the angular position: $y = K\varphi$, where $K = K_A m_0 g / k$ (V rad^{-1}). Using the sensor's output y as input in a proportional–derivative (PD) controller with feedback delay τ, the control torque is obtained as

$$Q = Py(t - \tau) + D\dot{y}(t - \tau) = PK\varphi(t - \tau) + DK\dot{\varphi}(t - \tau), \tag{2.3}$$

where P (N V^{-1}) and D (N s V^{-1}) are the proportional and the differential control gains. In this case, the linearized equation of motion reads

$$J\ddot{\varphi}(t) - Hmg\varphi(t) = PK\varphi(t - \tau) + DK\dot{\varphi}(t - \tau). \tag{2.4}$$

This equation is one of the basic examples of RFDEs. The corresponding stability properties are well known and are presented in several textbooks (e.g. Kolmanovskii & Nosov 1986; Stepan 1989).

If the dynamic effects, namely, the angular acceleration of the rod, are also taken into account, then the displacement of the accelerometer reads

$$q = \frac{m_0 g}{k} \sin \varphi - \frac{m_0 L}{k} \ddot{\varphi}. \tag{2.5}$$

In this case, the linearized output is

$$y = K\varphi - K_1 \ddot{\varphi}, \tag{2.6}$$
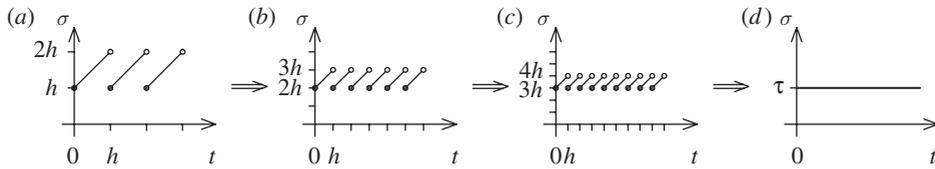
Figure 2. Sampling effect as time-periodic delay. (*a*) $r = 1$, (*b*) $r = 2$, (*c*) $r = 3$, and (*d*) $r \to \infty$.

where $K_1 = K_A m_0 L / k$ ($\mathrm{V\,s^2\,rad^{-1}}$) and K is the same as above. For this model, a PD controller with feedback delay τ results in the control torque

$$\begin{aligned} Q &= Py(t - \tau) + D\dot{y}(t - \tau) \\ &= PK\varphi(t - \tau) - PK_1\ddot{\varphi}(t - \tau) + DK\dot{\varphi}(t - \tau) - DK_1\dddot{\varphi}(t - \tau). \end{aligned} \tag{2.7}$$

In this case, the linearized equation of motion reads

$$J\ddot{\varphi}(t) - Hmg\varphi(t) = PK\varphi(t - \tau) - PK_1\ddot{\varphi}(t - \tau) + DK\dot{\varphi}(t - \tau) - DK_1\dddot{\varphi}(t - \tau). \tag{2.8}$$

This equation is now an AFDE, since the highest (third) derivative appears with a delayed argument only.

In this example, the advanced nature of the system comes from the fact that the accelerometer's signal is affected by the angular acceleration of the rod. Similarly to the previous example, mechanical systems occur that are governed by AFDEs via feeding back the acceleration signal using a PD controller with feedback delay, or by feeding back the jerk in a proportional (P) controller with feedback delay. Note that a similar case was investigated by Vyhlídal *et al.* (2009), where the velocity and the acceleration were fed back proportionally with feedback delay, resulting in an NFDE.

A kind of philosophical question arises. How can such systems be interpreted in practical applications, when the describing mathematical system is non-causal? Possible answers can be found in the small modelling inaccuracies that are neglected in the model, but they may have important qualitative effects on the behaviour of the system. Such a phenomenon is the sampling of digital control, which was not considered in equations (2.7) and (2.8). In digital control, the feedback is held constant for each sampling period h (ZOH), i.e. the control torque reads

$$\begin{aligned} Q &= Py(t_j - rh) + D\dot{y}(t_j - rh) \\ &= PK\varphi(t_j - rh) - PK_1\ddot{\varphi}(t_j - rh) + DK\dot{\varphi}(t_j - rh) - DK_1\dddot{\varphi}(t_j - rh), \\ & t \in [t_j, t_{j+1}), \end{aligned} \tag{2.9}$$

where $t_j = jh$, $j \in \mathbb{Z}$ and h is the sampling period such that $\tau = rh$ with r being an integer. Note that the sampled output can also be represented in the form

$$y(t_j - rh) = y(t - \sigma(t)), \quad t \in [t_j, t_{j+1}), \tag{2.10}$$

where

$$\sigma(t) = \tau - t_j + t, \quad t \in [t_j, t_{j+1}) \tag{2.11}$$
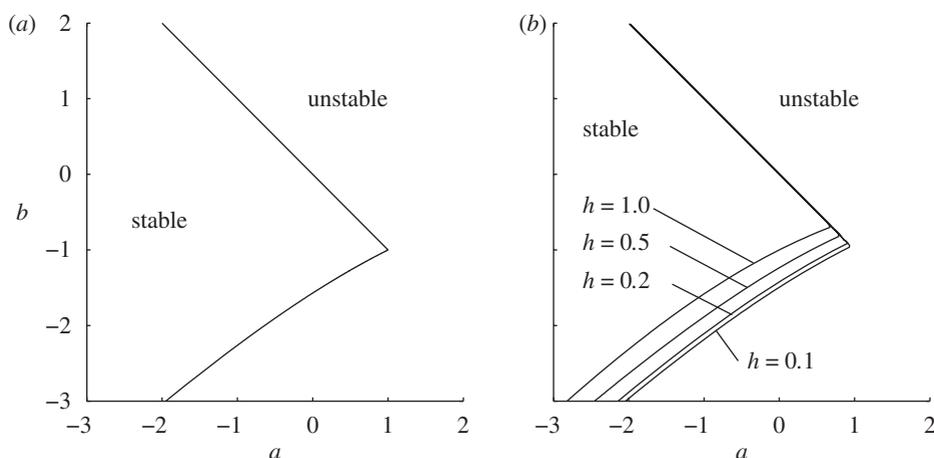
Figure 3. (*a*) Stability chart for the RFDE (3.1) for $\tau = 1$. (*b*) Stability charts for the sampled RFDE (3.2) for $\tau = 1$ and for different sampling periods h.

is a time-periodic time delay shown in figure 2. In fact, the sampling effect introduces a periodic parametric excitation at the time delay according to equation (2.11). It can be seen that the time-periodic time delay $\sigma(t)$ tends to the constant delay τ as the sampling period h tends to 0.

Note that the sampling effect presented above is equivalent to the zeroth-order semi-discretization method of RFDEs (see Insperger *et al.* 2008). Semi-discretization is a discretization technique where only the terms with delayed arguments are discretized while the other non-delayed terms are left in their original form. It is known that discretization techniques preserve exponential stability for RFDEs (Győri *et al.* 1998). However, for NFDEs, this is not true in general, as shown by Fabiano & Turi (1999). Preservation of exponential stability of NFDEs under delay perturbations was also analysed by Győri & Hartung (2001). Here, it will be shown for AFDEs that the discretized (or sampled) system can be stable in spite of the fact that the original AFDE is unstable with infinitely many unstable poles.

3. Delayed, neutral and advanced functional differential equations

In this section, first-order FDEs of delayed, neutral and advanced types are considered with and without the sampling effect in the delayed term. Stability properties with respect to the system parameters are analysed.

(*a*) First-order delay–differential equation

Consider the first-order delay–differential equation

$$\dot{x}(t) = ax(t) + bx(t - \tau). \tag{3.1}$$

This equation is a special type of Cushing's equation (Cushing 1977) and it is often referred to as one of the simplest basic examples for a delayed system (e.g. Stepan 1989; Hale & Lunel 1993; Michiels *et al.* 2002). The stability condition for the parameters a and b was first presented by Hayes (1950). The corresponding stability chart can be seen in figure 3*a*.

Equation (3.1) can also be considered as a control system with delayed feedback of the state, where a is the system parameter, b is the control gain and τ is the feedback delay. In the case of digital control, ZOH arises in the feedback loop, thus the argument of the delayed term becomes piecewise constant. The corresponding equation reads

$$\dot{x}(t) = ax(t) + bx(t_j - rh), \quad t \in [t_j, t_{j+1}), \tag{3.2}$$

where h is the sampling period, $t_j = jh$ and $j, r \in \mathbb{Z}$ such that $rh = \tau$. Owing to the piecewise constant argument of the delayed term, equation (3.2) can be solved over each sampling period $[t_j, t_{j+1})$, and the finite-dimensional discrete map can be constructed

$$y_{j+1} = \boldsymbol{\Phi} y_j, \tag{3.3}$$

where

$$y_j = \begin{pmatrix} x(t_j) \\ x(t_{j-1}) \\ x(t_{j-2}) \\ \vdots \\ x(t_{j-r}) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Phi} = \begin{pmatrix} \mathrm{e}^{ah} & 0 & \ldots & 0 & \dfrac{b}{a}(\mathrm{e}^{ah} - 1) \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{3.4}$$

The system is asymptotically stable if the eigenvalues of the monodromy matrix $\boldsymbol{\Phi}$ are in modulus less than 1. The corresponding stability charts can be seen in figure 3b for $\tau = 1$ and for different sampling periods. It can be seen that, by decreasing the sampling period h while keeping $rh = \tau = 1$ constant, the stability boundaries converge to that of equation (3.1). This is due to the fact that (3.2) is the zeroth-order semi-discretization of equation (3.1) according to Insperger *et al.* (2008). Since semi-discretization preserves asymptotic stability of the original equation (3.1) (see Hartung *et al.* 2006), it can be used to obtain approximations of the stability charts, as demonstrated in figure 3.

(b) First-order differential equation of neutral type

Consider the first-order differential equation of neutral type

$$\dot{x}(t) = ax(t) + b\dot{x}(t - \tau). \tag{3.5}$$

This equation can be considered as a control system with delayed feedback of the state's derivative. The condition for asymptotic stability is $a < 0$ and $|b| < 1$ (see Kolmanovskii & Nosov 1986). The corresponding stability chart can be seen in figure 4a.

The sampled counterpart of equation (3.5) reads

$$\dot{x}(t) = ax(t) + b\dot{x}(t_j - rh), \quad t \in [t_j, t_{j+1}). \tag{3.6}$$

Similarly to equation (3.2), equation (3.6) can be transformed into a discrete map of the form

$$z_{j+1} = \boldsymbol{\Psi} z_j, \tag{3.7}$$
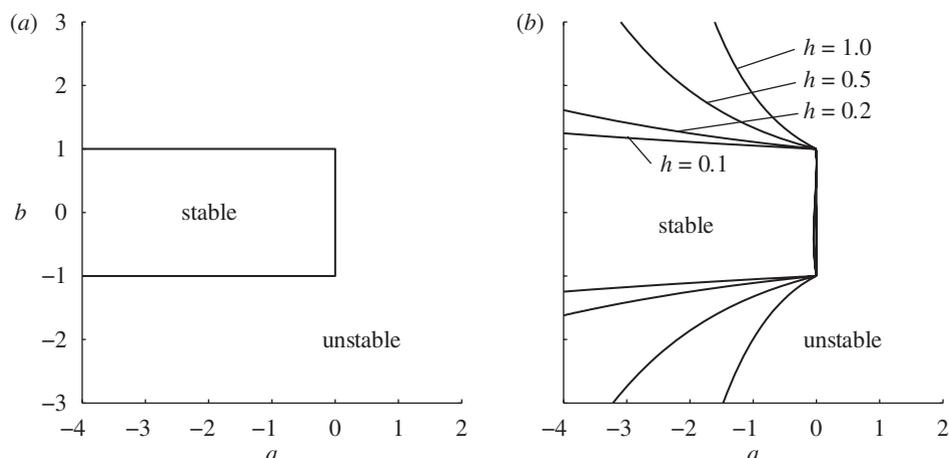
Figure 4. (*a*) Stability chart for the NFDE (3.5) for $\tau = 1$. (*b*) Stability charts for the sampled NFDE (3.6) for $\tau = 1$ and for different sampling periods h.

where

$$
z_j = \begin{pmatrix} x(t_j) \\ \dot{x}(t_j) \\ \dot{x}(t_{j-1}) \\ \vdots \\ \dot{x}(t_{j-r}) \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \mathrm{e}^{ah} & 0 & \cdots & 0 & \dfrac{b}{a}(\mathrm{e}^{ah} - 1) \\ a\mathrm{e}^{ah} & 0 & \cdots & 0 & b\mathrm{e}^{ah} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{3.8}
$$

The system is asymptotically stable if the eigenvalues of the monodromy matrix Ψ are in modulus less than 1. The corresponding stability charts can be seen in figure 4*b* for $\tau = 1$ and for different sampling periods. The stability boundaries of the sampled system (3.6) seem to converge to that of the original (non-sampled) one (3.6). Note that, although the stability charts of the sampled system in figure 4 seem to converge to the chart of equation (3.5), the above discretization does not necessarily preserve exponential stability for NFDEs generally (Fabiano & Turi 1999).

(*c*) *First-order differential equation of advanced type*

Consider the first-order differential equation of advanced type

$$
\dot{x}(t) = ax(t) + b\ddot{x}(t - \tau). \tag{3.9}
$$

This system is asymptotically stable for $b = 0$ and $a < 0$, and is unstable for any other parameter combinations (see Èl'sgol'c 1964). If $b \neq 0$, then the system has infinitely many unstable poles (i.e. roots of the characteristic equation with positive real parts).

The sampled counterpart of equation (3.9) reads

$$
\dot{x}(t) = ax(t) + b\ddot{x}(t_j - rh), \quad t \in [t_j, t_{j+1}). \tag{3.10}
$$
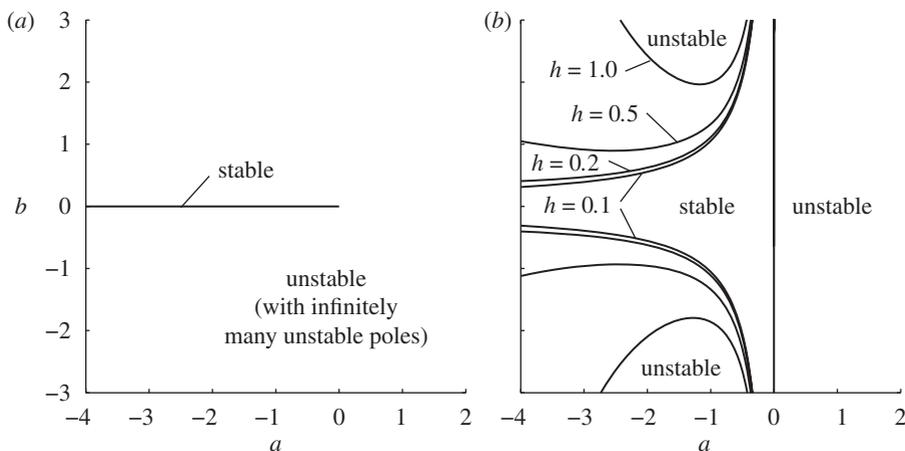
Figure 5. (*a*) Stability chart for the AFDE (3.9) for $\tau = 1$. (*b*) Stability charts for the sampled AFDE (3.10) for $\tau = 1$ and for different sampling periods h.

This is an equation of advanced type; again, however, the argument of the advanced term is piecewise constant. Another form of equation (3.10) is

$$\dot{x}(t) = ax(t) + b\ddot{x}(t - \sigma(t)), \tag{3.11}$$

where $\sigma(t)$ is a time-periodic time delay defined in equation (2.11) and presented in figure 2. Similarly to the previous RFDE and NFDE cases, a discrete map can be constructed by step-by-step solution in the form

$$w_{j+1} = \Theta\, w_j, \tag{3.12}$$

where

$$w_j = \begin{pmatrix} x(t_j) \\ \ddot{x}(t_j) \\ \ddot{x}(t_{j-1}) \\ \vdots \\ \ddot{x}(t_{j-r}) \end{pmatrix} \quad \text{and} \quad \Theta = \begin{pmatrix} \mathrm{e}^{ah} & 0 & \dots & 0 & \dfrac{b}{a}(\mathrm{e}^{ah} - 1) \\ a^2\mathrm{e}^{ah} & 0 & \dots & 0 & abe^{ah} \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{3.13}$$

Again, this system is asymptotically stable if the eigenvalues of the monodromy matrix Θ are in modulus less than 1. The corresponding stability charts can be seen in figure 5*b* for $\tau = 1$ and for different sampling periods. It can be seen that while the original equation (3.9) is always unstable for $b \neq 0$ with infinitely many unstable roots, the sampled system can be stable even for $b \neq 0$. Furthermore, it can be seen that the stability boundaries do not converge to the boundaries of the non-sampled system.

4. Advanced functional differential equation stability diagrams

In order to obtain an insight into the stability properties of equation (3.10), the eigenvalues of the corresponding monodromy matrix Θ should be analysed in detail. For a given sampling period h, the matrix Θ has $r + 2 = \tau/h + 2$
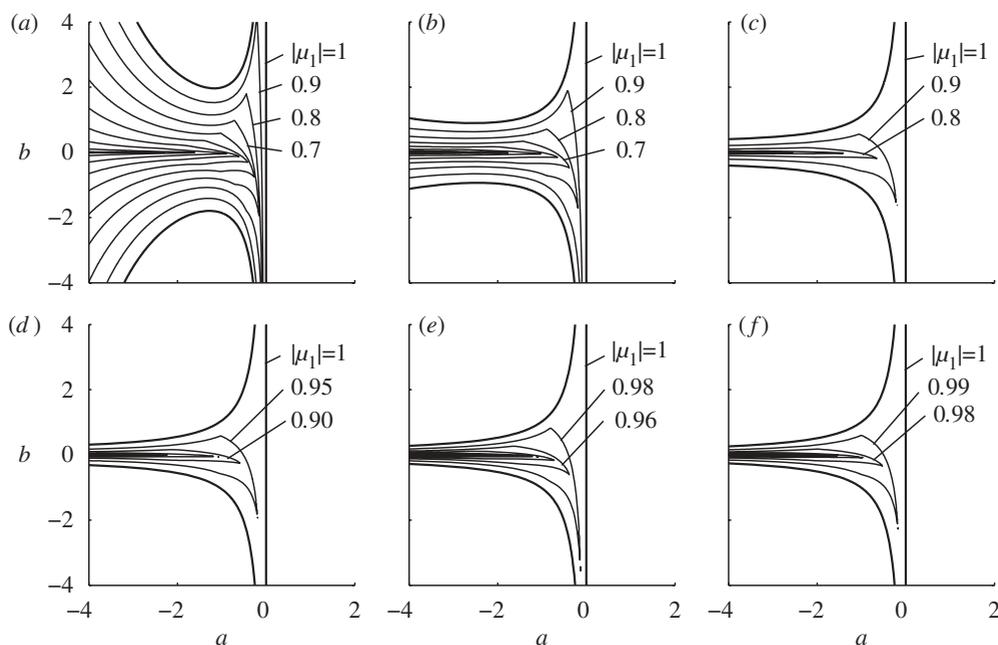
Figure 6. Contour plots for the critical eigenvalues of the sampled AFDE (3.10) for $\tau = 1$ and for different sampling periods h. (a) $h = 1$, (b) $h = 0.5$, (c) $h = 0.2$, (d) $h = 0.1$, (e) $h = 0.05$, and (f) $h = 0.02$.

eigenvalues that can be computed numerically. The magnitude $|\mu_1|$ of the critical (maximum in modulus) eigenvalue μ_1 characterizes the decay of the system's oscillations for perturbations, i.e. $\|w_{j+1}\| \leq |\mu_1| \|w_j\|$. Figure 6 shows the contour plots of the maximum eigenvalues for different sampling periods. Stability boundaries are denoted by the contour lines where $|\mu_1| = 1$ (these are denoted by thick lines). It can be seen that the stable domains in the stability diagrams do not disappear as the sampling period decreases; however, the magnitude of the critical eigenvalue within the domain of stability approaches 1 (see the contour lines and the associated modulus numbers).

Figure 7 shows the number of unstable poles for system (3.10). The stable domains are denoted by 0. It can be seen that, as the sampling period h is decreased, i.e. as the size of the monodromy matrix is increased, the number of unstable poles increases in the unstable domains. The number of unstable poles is in the range of $\tau/h = r$.

5. Numerical analysis of the transition $h \to 0$

In order to analyse the transition between the sampled AFDE (3.10) and the original AFDE (3.9), stability diagrams in the plane of h and b are constructed for fixed negative a. These diagrams can be determined numerically by increasing the parameter b step by step until the magnitude of the critical eigenvalue becomes larger than 1. This gives the stability boundaries for positive b. Similarly, the stability boundaries for negative b are obtained by decreasing b step by step.
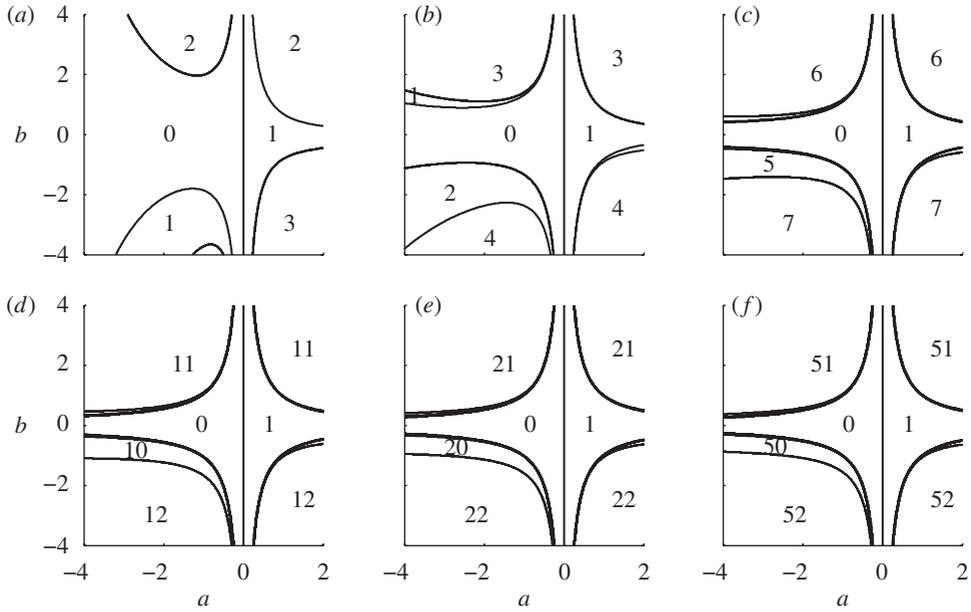
Figure 7. Number of unstable eigenvalues of the sampled AFDE (3.10) for $\tau = 1$ and for different sampling periods h. (a) $h = 1$, (b) $h = 0.5$, (c) $h = 0.2$, (d) $h = 0.1$, (e) $h = 0.05$, and (f) $h = 0.02$.
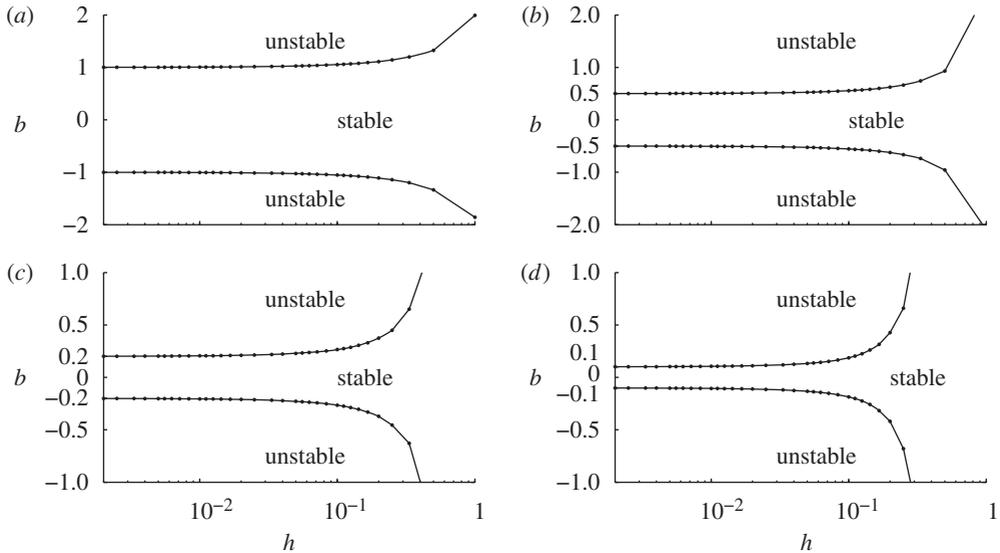
Figure 8. Stability domains for the sampled AFDE (3.10) in the plane (h, b) for $\tau = 1$ and for different system parameters a. (a) $a = -1$, (b) $a = -2$, (c) $a = -5$, and (d) $a = -10$.

Figure 8 presents these diagrams for $a = -1$, -2, -5 and -10. These numerical results show that the stability domain does not disappear as the sampling period h tends to zero. Furthermore, they suggest that the critical value of b converges to $\pm 1/a$. This latter feature can easily be confirmed by writing the characteristic
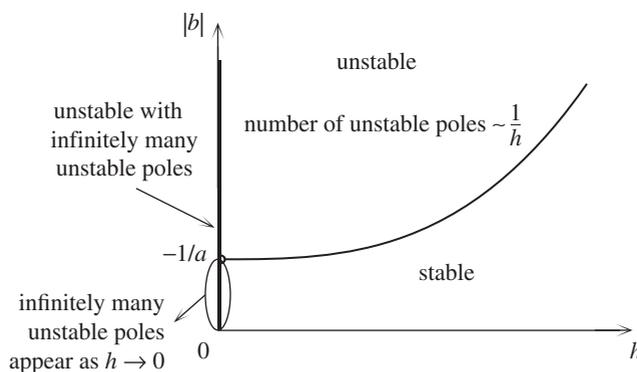
Figure 9. Sketch of the stability properties of the sampled AFDE (3.10) in the plane (h, b) for fixed $a < 0$.

equation of the monodromy matrix Θ in the form

$$(\mu - e^{ah})(\mu^{r+1} - abe^{ah}) = abe^{ah}(e^{ah} - 1). \tag{5.1}$$

For a small sampling period h, the roots can be approximated as

$$\mu_1 \approx 1 + ah \quad \text{and} \quad \mu_l \approx [ab(1 + ah)]^{1/(r+1)} e^{i2l\pi/(r+1)}, \quad l = 2, 3, \ldots, r + 2. \tag{5.2}$$

The condition for asymptotic stability in this case is $a < 0$ with $|ab(1 + ah)| < 1$. This confirms that, as h tends to zero, the stability boundaries in the plane (a, b) tend to $b = \pm 1/a$ with $a < 0$. On the other hand, the transition $h \to 0$ corresponds to the original AFDE, i.e. in this case the system has infinitely many unstable poles if $h = 0$ and $b \neq 0$. This shows that, as $h \to 0$, the system is stable if $|b| < -1/a$ with $a < 0$, but in the limit case, i.e. when $h = 0$, 'suddenly' infinitely many poles become unstable. This phenomenon is presented in figure 9.

The approximations in equation (5.2) show that, as the sampling period h gets smaller and smaller, one pole (μ_1) approaches to 1, while the other ($r + 1$) poles cluster round the circle of radius ab in the complex plane. This phenomenon is presented in figure 10 for some special parameter combinations. Three special cases are considered:

— $a = -1$, $b = 2.5$: The system is unstable for any sampling period h. The number of unstable poles increases as h decreases. For $h < 0.2$, the magnitude of the critical eigenvalue is decreasing, and the poles approach the unit circle from outside.

— $a = -1$, $b = 1.5$: The system is stable for $h = 1$ but unstable for any other $h < 1$. The number of unstable poles increases as h decreases. The magnitude of the critical eigenvalue is decreasing, and the poles approach the unit circle from outside.

— $a = -1$, $b = 0.5$: The system is stable for all $h > 0$. The number of stable poles increases as h decreases. The magnitude of the critical eigenvalue is increasing, and the poles approach the unit circle from inside.
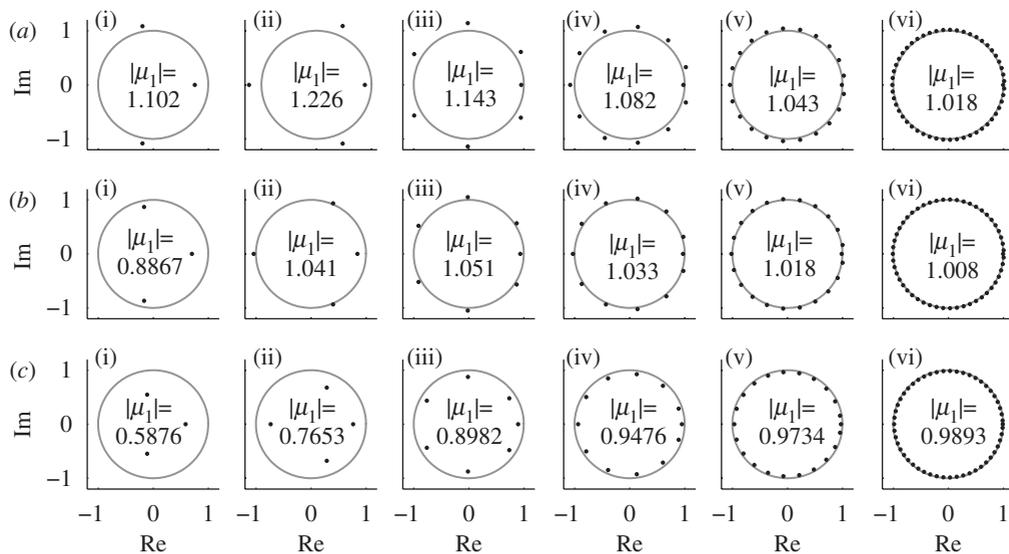
Figure 10. Location of the poles for different parameter combinations. The unit circle is denoted by the grey line. (a) $a = -1, b = 2.5$, (b) $a = -1, b = 1.5$, and (c) $a = -1, b = 0.5$. (i) $h = 1$, (ii) $h = 0.5$, (iii) $h = 0.2$, (iv) $h = 0.1$, (v) $h = 0.05$, and (vi) $h = 0.02$.

The magnitude $|\mu_1|$ of the critical eigenvalue is given for each plot. These diagrams demonstrate how the jump of infinitely many poles outside the unit circle appears as $h \to 0$.

6. Conclusion

In this paper, the AFDE (3.9) and its sampled counterpart (3.10) or, in other form, equation (3.11), were analysed. It was shown that the sampled AFDE (3.10) is asymptotically stable for certain parameter combinations while the original AFDE (3.9) is always unstable with infinitely many unstable poles (if $b \neq 0$). The transition $h \to 0$ was analysed and it was found that the stable parameter domains of the sampled AFDE do not disappear as $h \to 0$, in spite of the fact that, in the limit case when $h = 0$, the system has infinitely many unstable poles. This unusual phenomenon, namely, that infinitely many unstable poles appear for an infinitesimal change in the system's parameters, is not unique for FDEs—see, for example, the case of a delayed follower force in example 3.7 in Stepan (1989).

From a practical point of view, this strange transition between the sampled and the continuous cases is not very interesting, since as $h \to 0$, more and more eigenvalues get closer and closer to the unit circle. For instance, a system with many critical poles of real parts $\ln 0.99$ can be considered practically unstable, since the decay of the oscillations is extremely slow compared with the dynamics of the system.

However, the interesting point is that an advanced system with delayed feedback of the highest derivative (which is known as infinitely unstable) can be stabilized by periodic sampling of the feedback, i.e. by making the argument

of the delayed term piecewise constant. This phenomenon is somewhat similar to the delayed Mathieu equation (Insperger & Stepan 2002), where a parametric excitation in the stiffness stabilizes the delayed system. In the current case, sampling can be considered as a kind of parametric excitation in the delay as shown in equation (3.11) with equation (2.11) (see also figure 2). In this case, parametric excitation in the delay stabilizes the otherwise infinitely unstable system.

In the current analysis, one of the simplest AFDEs was considered; however, similar equations may appear in many realistic problems, like in the case of the stick balancing problem introduced in §2. In general, equations with advanced terms can typically arise in controlled mechanical systems, since the output may easily be affected by the acceleration or by the jerk of the system. This effect is often considered to be a kind of noise in the output, and is therefore often neglected; however, it may have an important role in the system's dynamics. Consider the output of a general system in the form

$$y(t) = f(t, x(t - \tau), \varepsilon_1 \dot{x}(t - \tau), \varepsilon_2 \ddot{x}(t - \tau)), \qquad (6.1)$$

where $y \in \mathbb{R}^m$ is the output, $x \in \mathbb{R}^n$ is the state, and ε_1 and ε_2 are small, but non-zero, parameters. Usually, such effects are neglected, i.e. ε_1 and ε_2 are considered to be zero. However, if they are not neglected, then the linearized equation has the form

$$\dot{x}(t) = Ax(t) + B_0 x(t - \tau) + \varepsilon_1 B_1 \dot{x}(t - \tau) + \varepsilon_2 B_2 \ddot{x}(t - \tau) \qquad (6.2)$$

with A, B_0, B_1 and B_2 being appropriate matrices obtained during the linearization. If the parameter ε_2 is non-zero, then this equation is an AFDE, and the system is unstable with infinitely many unstable poles, independently of the exact value of ε_2. While the effect of higher-order derivatives is rarely modelled in practical applications, a system's unexpected instability or its other weird behaviour is often referred to as the consequence of 'unmodelled system dynamics'. If the system works properly then it is often attributed to the robustness of the controller and often no detailed explanation is given. In this work, it was shown that one explanation for the stable operation of such systems may be the sampling effect of digital control.

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