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Micro-Chaos in Digital Control

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Summary. In this paper we analyze a model for the effect of digital control on one-dimensional, linearly unstable dynamical systems. Our goal is to explain the existence of small, irregular oscillations that are frequently observed near the desired equilibrium. We derive a one-dimensional map that captures exactly the dynamics of the continuous system. Using this *micro-chaos map*, we prove the existence of a hyperbolic strange attractor for a large set of parameter values. We also construct an "instability chart" on the parameter plane to describe how the size and structure of the chaotic attractor changes as the parameters are varied. The applications of our results include the stick-and-slip motion of machine tools and other mechanical problems with locally negative dissipation.

1. Introduction

Consider the near-equilibrium motion of a one-degree-of-freedom mechanical system under the effect of velocity-dependent forces. In particular, assume that the system is subject to some *negative* velocity-dependent dissipation or an accelerating force which is linear in the velocity. Then, in nondimensionalized form, the velocity v satisfies the linear differential equation

$$\dot{v} - kv = 0,$$

with $k > 0$. We want to counteract the effect of the force kv by introducing a computer-controlled dissipation term which is linear in velocity. Ideally, such a force would change the above equation to

$$\dot{v} - kv = -pv,$$

where $p > k$ is the damping coefficient. However, the control we use is assumed to have two deficiencies. First, the computer samples the velocity only at discrete time instances with sampling time $\Delta t > 0$. As a result, the dissipative force applied by the control

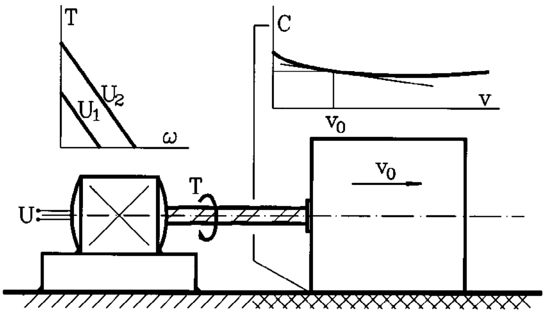

Fig. 1. The control of stick-and-slip motion.

system would be $-pv(j\Delta t)$ throughout the time interval $[j\Delta t, (j+1)\Delta t)$, where j is a positive integer. Second, the velocity measurement has a finite resolution, i.e., velocity is measured by the system in terms of the multiples of some small velocity unit $h > 0$. This implies that for $t \in [j\Delta t, (j+1)\Delta t)$ the actual force applied by the control system will be $-ph\,\mathrm{Int}\,(v(j\Delta t)/h)$. Introducing the notation $t_j = j\Delta t$, we arrive at the following equation for the velocity v:

$$\dot{v}(t) - kv(t) = -ph\,\mathrm{Int}\left(\frac{v(t_j)}{h}\right), \qquad t \in [t_j, t_{j+1}). \tag{1.1}$$

This equation arises, e.g., in the study of stick-and-slip motion of certain machine tool parts (see, e.g., [15]). For these systems digital control is used to achieve a stable, small feed rate for the tool. The corresponding mechanical model consists of a block sliding on a surface near some prescribed velocity v_0 under the action of an electric motor (see Figure 1). At low speed the combined dry and viscous friction force C acting on the block is locally decreasing as the velocity v increases. The electric motor introduces a dissipative force described by the torque-speed $(T - \omega)$ characteristics of the motor, but the system may still be unstable at v_0. In that case, an additional control force provided by the electric motor (with input voltage U) is used to keep the velocity v_0 stable. This introduces artificial dissipation which increases linearly with the velocity.

One finds that for $t \in [t_j, t_{j+1})$, (1.1) admits the solution

$$v(t) = \left(v_j - \frac{ph}{k}\,\mathrm{Int}\left(\frac{v_j}{h}\right)\right)e^{k(t-t_j)} + \frac{ph}{k}\,\mathrm{Int}\left(\frac{v_j}{h}\right), \tag{1.2}$$

where $v_j = v(t_j)$. From (1.2) we directly obtain that

$$v_{j+1} = \lim_{t\to t_{j+1}} v(t) = \left(v_j - \frac{ph}{k}\,\mathrm{Int}\left(\frac{v_j}{h}\right)\right)e^{k\Delta t} + \frac{ph}{k}\,\mathrm{Int}\left(\frac{v_j}{h}\right), \tag{1.3}$$

or

$$v_{j+1} = v_j e^{k\Delta t} - \frac{p}{k}(e^{k\Delta t} - 1)h\,\mathrm{Int}\left(\frac{v_j}{h}\right). \tag{1.4}$$

Let us introduce the parameters

$$a = e^{k\Delta t} > 1, \qquad b = \frac{p}{k}(e^{k\Delta t} - 1) = \frac{p}{k}(a-1) > a - 1.$$

Equation (1.4) shows that the velocity values at the time instances $j\,\Delta t$ can be obtained by iterating the one-dimensional mapping

$$x \mapsto ax - bh\,\mathrm{Int}(x/h), \tag{1.5}$$

where the initial value for the iteration is $x_0 = v_0$, the velocity at some time t_0.

This mapping and its multidimensional analogs have a central role in describing the local dynamics of digitally controlled systems. When a processor is used to stabilize the unstable equilibrium of a mechanical system, the sampling delay and the round-off errors at the analog-digital converters frequently result in small amplitude stochastic vibrations around the desired equilibrium. Such problems were considered by Ushio and Hsu [17], who studied the dynamics of a corresponding two-dimensional map. Delchamps [6] formulated the general control problem of an n-dimensional, discrete linear system and analyzed the $n = 1$ case in more detail (see Section 6 for a comparison with our results). Stépán [13] and Enikov and Stépán [8] studied analytically and experimentally the small amplitude stochastic motion of an inverted pendulum attached to a moving cart. In that example digital control was used to stabilize the upright position of the pendulum. In linear approximation the corresponding discrete control problem can be described by a three-dimensional map of the same form as (1.5). Among practical engineering applications, precision control tasks appear to be the most important ones. An example can be found, e.g., in the paper of Ueda et al. [16] on the machining of mirrors.

Although most of these problems are multidimensional, even the study of the general one-dimensional digital control problem has been missing in the literature. Our goal in this paper is to provide a detailed analysis of the one-dimensional case described by the map (1.5). We prove the existence of a chaotic attractor for the system and identify parameter domains with the same type of chaotic dynamics. By "same type" we mean identical symbolic dynamics. We also present estimates for some characteristic quantities like amplitude and frequency range, entropy, and fractal dimension. We believe that our results are of substantial practical importance and may be used to improve the design of digitally controlled systems.

2. Notation and Definitions

To give a general formulation of our problem, let us consider a map of the form (1.5) with

$$(a, b) \in P = \{(\alpha, \beta) \in \mathbb{R}^2 \mid 0 < \alpha - 1 < \beta < \alpha\}.$$

For convenience, we introduce the rescaling $x \to x/h$ to obtain the equivalent map (with $m = \mathrm{Int}(1/h) + 1$):

$$\begin{aligned} &F\colon [0, m] \to I = [0, m],\\ &x \mapsto ax - b\,\mathrm{Int}(x), \end{aligned} \tag{2.1}$$

which we shall refer to as the *μ-chaos map*. Note that F is a piecewise linear, monotone, upper semicontinuous map with discontinuities at the integers $i = 1, 2, \ldots, m$. Over the interval $M_i = [i - 1, i)$, $F \equiv F_i$ can be written as

$$F_i(x) = ax - (i - 1)b, \qquad x \in M_i = [i - 1, i), \qquad i = 1, \ldots, m. \tag{2.2}$$

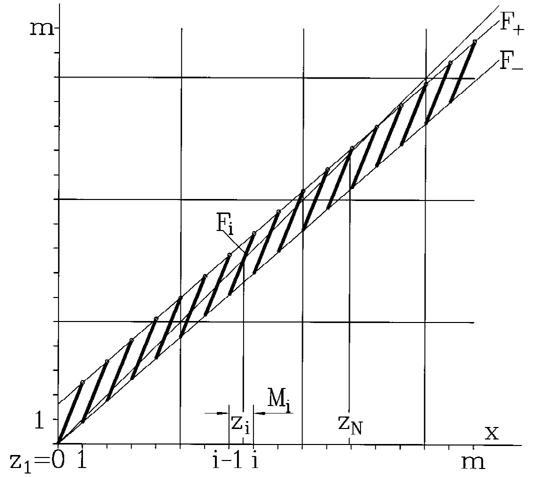

Fig. 2. Graph of F for the parameter values $a = 5/2$, $b = 13/8$.

We will also need the auxiliary maps $F_-, F_+ \colon I \to I$ defined as

$$F_-(x) = (a-b)x, \qquad F_+(x) = (a-b)x + b. \tag{2.3}$$

A sketch of the graphs of F, F_-, and F_+ can be seen in Figure 2. Note that

$$F_-(x) \le F(x) < F_+(x), \qquad x \in I. \tag{2.4}$$

It is easy to verify that F has N fixed points given by

$$z_i = \frac{b(i-1)}{a-1}, \qquad i = 1, 2, \ldots, N, \tag{2.5}$$

with N defined as

$$N = \max_{i \in Z^+} \left\{ i \mid i < \frac{b}{b+1-a} \right\}. \tag{2.6}$$

We remark that F is only *upper* semicontinuous; that is why we did not simply define N as the integer part of $b/(b+1-a)$. As one immediately sees from Figure 2, all the fixed points z_i are unstable. In the plane of the parameters a and b the region $Z_j \subset P$ yielding exactly j fixed points is given by

$$Z_j = \{(a, b) \in P \mid c_{j+1}(a) \le b < c_j(a)\}, \tag{2.7}$$

where the lines c_j are defined as

$$b = c_j(a) = \frac{j(a-1)}{j-1}. \tag{2.8}$$

We show some of these domains in Figure 3.

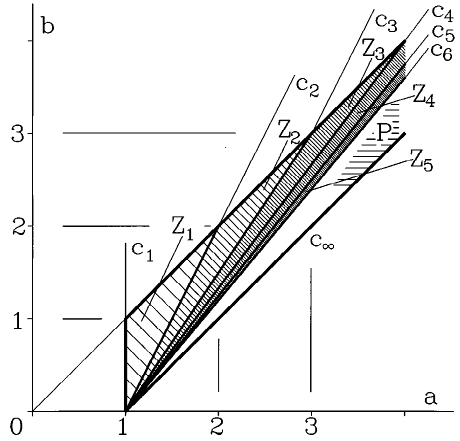

Fig. 3. Parameter domains Z_j referring to j fixed points.

Our program for the study of F is as follows. In Section 3 we identify a positively invariant set $\mathcal{A}$ for the map F and show that it is in fact a hyperbolic strange attractor. In Section 4 we consider a Cantor set Λ within $\mathcal{A}$ which is responsible for the "strange" behavior in the attractor. We describe Λ in symbolic dynamics terms. We also study the domain of attraction $\mathcal{D}$ of $\mathcal{A}$ and present symbolic dynamics for certain hyperbolic chaotic sets not lying in $\mathcal{D}$. In Section 5 we give a topological characterization of the Cantor set Λ. Finally, we summarize our results and relate them to previous work on one-dimensional digital control.

3. The Invariant Set $\mathcal{A}$ and Its Properties

Before we start the main topic of this section, we make our first observation on the map F which suggests irregular features in its dynamics.

Proposition 3.1. *F has sensitive dependence on initial conditions.*

Proof. Let us fix the constant

$$\delta = \frac{b}{a+1}.$$

We will show that for any $x_1 \neq x_2$ with $|x_2 - x_1| < \delta$, there exists $N \geq 1$ such that

$$|F^N(x_2) - F^N(x_1)| \geq \delta. \tag{3.1}$$

Since F expands the distances of points taken from the same interval M_i, without loss of generality we can assume that after n iterations

$$F^n(x_1) \in M_i, \qquad F^n(x_2) \in M_j, \qquad i \neq j. \tag{3.2}$$

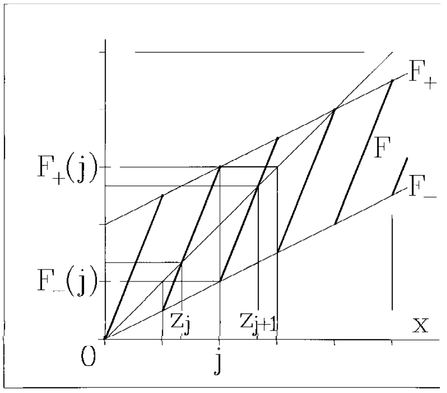

Fig. 4. Condition $H(j)$ in case of $a = 2$, $b = 5/2$, $j = 2$.

If $|F^n(x_2) - F^n(x_1)| \geq \delta$, we are done. If not, then we have

$$|F^n(x_2) - F^n(x_1)| < \delta = \frac{b}{a+1} < 1,$$

and hence, supposing $F^n(x_2) > F^n(x_1)$, we must have $j = i+1$ in (3.2). Then, recalling the definition of the map F_i from (2.2), we can write

$$\begin{aligned} |F^{n+1}(x_2) - F^{n+1}(x_1)| &= |F_{i+1} \circ F^n(x_2) - F_i \circ F^n(x_1)| \\ &= |a(F^n(x_2) - F^n(x_1)) - b| \\ &\geq b - a|F^n(x_2) - F^n(x_1)| > b - a\delta = \delta; \end{aligned}$$

hence the choice $N = n + 1$ completes the proof. □

In what follows we will be interested in parameter configurations for the map F for which there exists a positive integer $2 \leq j \leq m - 1$ such that the following condition $H(j)$ is satisfied:

$H(j)$ (i) $F_+(j) > z_{j+1}$,
(ii) $F_-(j) < z_j$.

The geometric meaning of condition $H(j)$ can be seen in Figure 4. Note that if $H(j)$ holds, then

$$\begin{aligned} F([z_j, j)) &\supset [z_j, j] \cup [j, z_{j+1}], \\ F([j, z_{j+1}]) &\supset [z_j, j] \cup [j, z_{j+1}]. \end{aligned} \tag{3.3}$$

Also observe that if we require equalities in condition $H(j)$, then the interval $[z_j, j] \cup [j, z_{j+1}] = [z_j, z_{j+1}]$ is invariant under F, as shown in Figure 5. If we require equality in (i) of $H(j)$ and adapt (ii) as above, then any iterate of a point $x \in [z_j, z_{j+1}]$ can only leave this interval towards smaller x values, i.e., to the left. On the other hand, if (i) holds

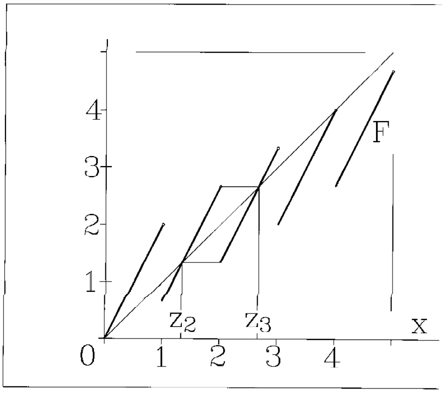

Fig. 5. The invariant interval $[z_j, z_{j+1}]$ in case of $a = 2$, $b = 4/3$, $j = 2$.

as above but we require equality in (ii), then iterates of x can only leave the interval $x \in [z_j, z_{j+1}]$ to the right.

As we shall see later (and as one can immediately guess from (3.3)), $H(j)$ implies the existence of a *horseshoe* within the interval $[z_j, z_{j+1}]$. By "horseshoe" here we mean an invariant Cantor set on which F is topologically conjugate to a (one-sided) Bernoulli-shift on two symbols, i.e., it shares the symbolic dynamics of the Smale-horseshoe map (see, e.g., [9]). In our case one may assign a symbol to the n-th iterate $F^n(x)$ of $x \in [z_j, z_{j+1}]$ based on a partition of the interval I that will be constructed later in this section.

As a first step in our analysis, we now rephrase condition $H(j)$ in terms of the parameters a and b.

Proposition 3.2. *Condition $H(j)$ above is equivalent to*

(i)

$$b < d_j^+(a) = \frac{ja(a-1)}{1 + (j-1)a}, \tag{3.4}$$

(ii)

$$b > d_j^-(a) = \frac{ja(a-1)}{aj - 1}. \tag{3.5}$$

Proof. From (2.3) and (2.5) we obtain that condition $H(j)$ is equivalent to

$$(a-b)j + b > \frac{jb}{a-1}, \tag{3.6}$$

$$(a-b)j < \frac{(j-1)b}{a-1}. \tag{3.7}$$

These inequalities and the fact that $(a, b) \in P$ imply the statement of the proposition. □

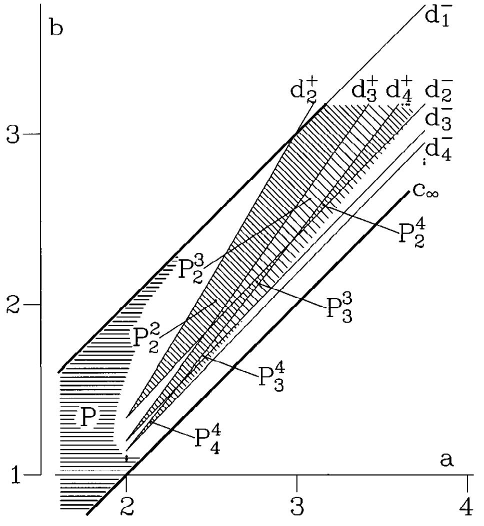

Fig. 6. The parameter domains $P_{j_1}^{j_2}$.

Proposition 2.2 enables us to identify regions in the (a, b) parameter plane in which $H(j)$ holds (which requires $j \geq 2$, and hence $m \geq 3$, as one can easily check). Let us fix two integers j_1 and j_2 with $2 \leq j_1 \leq j_2$, and define the subset $P_{j_1}^{j_2}$ of the parameter space P as

$$P_{j_1}^{j_2} = \{(a, b) \in P \mid b \in (d_{j_1}^-(a), d_{j_1-1}^-(a)] \cap [d_{j_2+1}^+(a), d_{j_2}^+(a))\}. \tag{3.8}$$

Note that for any $(a, b) \in P_{j_1}^{j_2}$, $j = j_1$ and $j = j_2$ are the minimal and maximal integers, respectively, for which condition $H(j)$ holds. In Figure 6 we plot some of the sets $P_{j_1}^{j_2}$ on the parameter plane (a, b). Note that the graphs of d_j^+ and d_j^- intersect at $(a, b) = (2, 2j/(2j - 1))$, which implies that all the sets P_j^j lie in the $a > 2$ open half-plane of the parameter plane. Similar calculation shows that the sets $P_{j_1}^{j_2}$ also lie in the $a > (j_1 + j_2)/j_1$ half-plane. Any parameter point in P with $a > 3$ belongs to one of the domains $P_{j_1}^{j_2}$. As far as the shape of these regions is concerned, note that the graph of d_j^+ is asymptotic from below to the graph of c_j, while the graph of d_j^- is asymptotic from above to the line c_∞ (i.e., to $b = a - 1$; see Figure 6). Furthermore, we have the relations

$$a < (>)\, \frac{j_1 + j_2 + 1}{j_1} \;\Rightarrow\; d_{j_1}^-(a) > (<)\, d_{j_2+1}^+(a),$$

$$a < (>)\, \frac{j_1 - 1 + j_2}{j_1 - 1} \;\Rightarrow\; d_{j_2}^+(a) < (>)\, d_{j_1-1}^-(a).$$

By Proposition 3.2, for $(a, b) \in P_{j_1}^{j_2}$, the set I contains $j_2 - j_1 + 1$ adjacent intervals of the form

$$L_j = [z_j, z_{j+1}], \qquad j = j_1, \ldots, j_2,$$

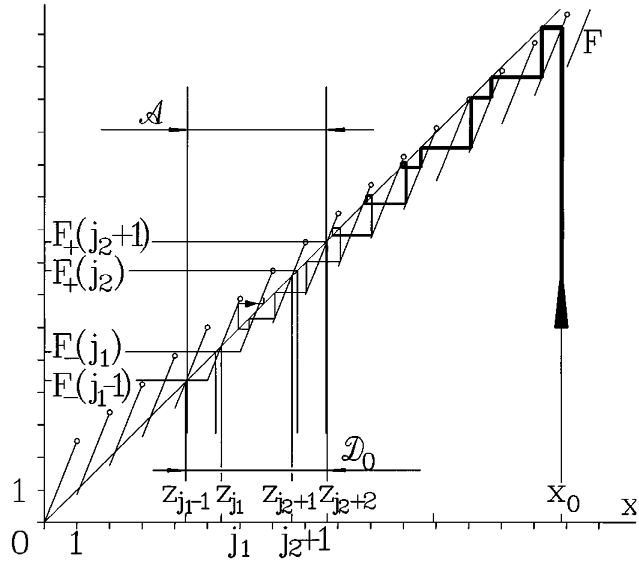

Fig. 7. The attractive set $\mathcal{A}$ in case of $a = 5/2$, $b = 13/8$, $j_1 = 6$, $j_2 = 7$.

such that on each of these intervals F admits a horseshoe-type dynamics (see formula (3.3) and Figure 5). The union of these intervals is usually not invariant under F; however, it is contained in the set

$$\mathcal{A} = [F_-(j_1 - 1), F_+(j_2 + 1)] = [(a - b)(j_1 - 1), (a - b)(j_2 + 1) + b], \qquad (3.9)$$

which will be of central importance to us. Figure 7 explains the construction of $\mathcal{A}$: One looks for the minimal invariant set which captures all the anticipated complicated dynamics of F, namely, the minimal invariant set containing all the intervals $[z_j, z_{j+1}]$ which are candidates for containing horseshoes. The following proposition makes this statement more precise.

Proposition 3.3. *$\mathcal{A}$ is a positively invariant attractive set.*

Proof. We first show that $\mathcal{A}$ is invariant under forward iterations of the map F. Let us observe that from the definition of F we have

$$x \in [j_1, j_2] \;\Rightarrow\; F(x) \in [F_-(j_1), F_+(j_2)). \qquad (3.10)$$

Since, by definition, for $(a, b) \in P_{j_1}^{j_2}$ the conditions $H(j_2 + 1)$ and $H(j_1 - 1)$ do not hold, we can write

$$\begin{aligned} x \in [F_-(j_1 - 1), j_1] \quad &\Rightarrow \quad F(x) \in [F_-(j_1 - 1), F_+(j_1)), \\ x \in [j_2, F_+(j_2 + 1)] \quad &\Rightarrow \quad F(x) \in [F_-(j_2), F_+(j_2 + 1)). \end{aligned} \qquad (3.11)$$

But (3.9), (3.10), and (3.11) imply $F(x) \in \mathcal{A}$ whenever $x \in \mathcal{A}$, and hence $\mathcal{A}$ is positively invariant under F.

We now show that $\mathcal{A}$ attracts iterates of any point taken from the open interval

$$\mathcal{D}_0 = (z_{j_1-1}, z_{j_2+2}) \supset \mathcal{A}. \tag{3.12}$$

Let us consider some $x_0 \in \mathcal{D}_0$ such that

$$x_0 \in (z_{j_1-1}, F_-(j_1-1)) \quad \Rightarrow x_0 < F(x_0) < F_+(j_1-1). \tag{3.13}$$

This shows that F has no fixed point in $(z_{j_1-1}, F_-(j_1-1)]$. Suppose that no iterate of x_0 falls in $\mathcal{A}$. Then, by (3.13) all the iterates of x_0 fall in $(z_{j_1-1}, F_-(j_1-1)]$ and form a bounded monotone sequence $\{F^n(x_0)\}_{n=0}^{\infty}$ with

$$\lim_{n\to\infty} F^n(x_0) = x^* \in (z_{j_1-1}, F_-(j_1-1)]. \tag{3.14}$$

Since on the interval $(z_{j_1-1}, F_-(j_1-1)]$ the map $F \equiv F_{j_1-1}$ is continuous, we must have

$$F(x^*) = x^*,$$

which contradicts the fact that F has no fixed point in $(z_{j_1-1}, F_-(j_1-1)]$. Hence we obtained that for any $x_0 \in (z_{j_1-1}, F_-(j_1-1))$, there exists $n_1 > 0$ such that $F^{n_1}(x_0) \in \mathcal{A}$. A similar argument shows that for any $x_0 \in (F_+(j_2+1), z_{j_2+2})$, there exists $n_2 > 0$ such that $F^{n_2}(x_0) \in \mathcal{A}$. Therefore, $\mathcal{D}_0$ is a subset of the domain of attraction $\mathcal{D}$ of the set $\mathcal{A}$. □

We remark that the above argument proves that $\mathcal{A}$ is globally attractive on $I - \{0\}$ if F has no fixed points outside $\mathcal{A} \cup \{0\}$. If F does have fixed points outside $\mathcal{A}$, then the full domain $\mathcal{D}$ of attraction of $\mathcal{A}$ is an open subset of $I - \mathcal{A}$ whose complement is a Cantor set. We will discuss this in Theorem 4.4 in Section 4.

To analyze the properties of the set $\mathcal{A}$ further, we introduce a partition of $\mathcal{A}$ into closed intervals in two steps. First, we extend the set of the intervals L_j we already used in the following way:

$$L_{j_1-1} = [F_-(j_1-1), z_{j_1}] = [(a-b)(j_1-1), \tfrac{b}{a-1}(j_1-1)],$$

$$L_j = [z_j, z_{j+1}] = [\tfrac{b}{a-1}(j-1), \tfrac{b}{a-1}j], \qquad j \in \{j_1, \dots, j_2\},$$

$$L_{j_2+1} = [z_{j_2+1}, F_+(j_2+1)] = [\tfrac{b}{a-1}j_2, b + (a-b)(j_2+1)].$$

To prepare a further partition of these intervals in the second step, let us define the nonnegative integers k_j^- and k_j^+ as

$$\begin{aligned} k_j^- &= \min\left\{k \in \mathbb{Z}^+ \;\mid\; F_j^{-k}(j) < F_+(j-1)\right\}, \qquad j \in \{j_1, j_1+1, \dots, j_2+1\}, \\ k_j^+ &= \min\left\{k \in \mathbb{Z}^+ \mid F_{j+1}^{-k}(j) > F_-(j+1)\right\}, \qquad j \in \{j_1-1, j_1, \dots, j_2\}. \end{aligned} \tag{3.15}$$

These integers are well-defined for parameter values $(a, b) \in P_{j_1}^{j_2}$, since

$$\lim_{k\to\infty} F_j^{-k}(j) = z_j < F_+(j-1), \qquad \text{for all } \; j \le j_2+1,$$

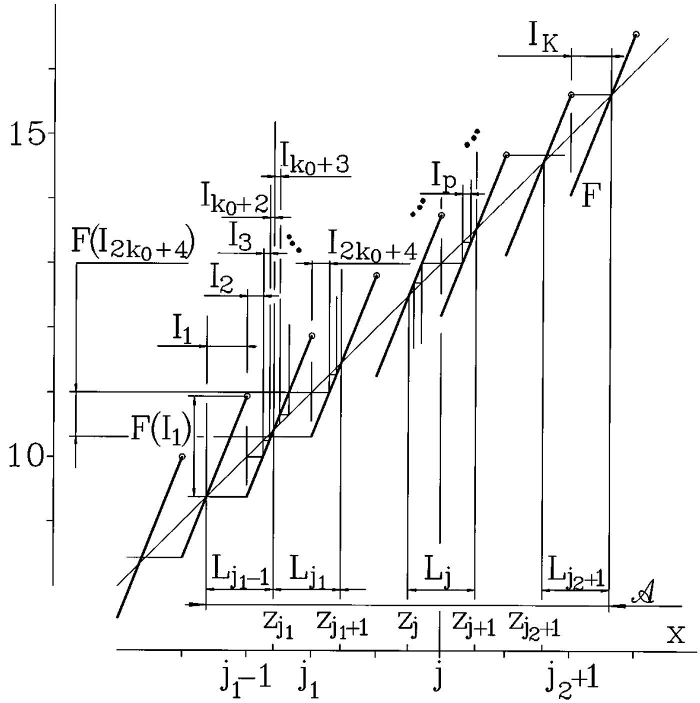

Fig. 8. Partition of $\mathcal{A}$ in case of $a = 5/2$, $b = 25/16$, $j_1 = 11$, $j_2 = 14$, $k_0 = 2$.

$$\lim_{k\to\infty} F_{j+1}^{-k}(j) = z_{j+1} > F_-(j+1), \qquad \text{for all } \; j \geq j_1 - 1.$$

In the above definitions, the "inverse" of F_j in (2.2) is defined as

$$F_j^{-1}(y) = \frac{1}{a}y + \frac{b}{a}(j-1), \qquad y \in [F_-(j-1), F_+(j)), \qquad j = j_1 - 1, \dots, j_2 + 1;$$

thus it is understood that $F_j^{-k} = F_j^{-(k-1)} \circ F_j^{-1}$ always projects into $[j-1, j)$ and $F_j^0 = I$. The latter formula means that the integers $k_j^{-,+}$ can also be zero if either $j < F_+(j-1)$ or $j > F_-(j+1)$.

Let us fix the nonnegative integer

$$k_0 = \max\left\{k_{j_1-1}^+, \dots, k_{j_2}^+, k_{j_1}^-, \dots, k_{j_2+1}^-\right\}. \tag{3.16}$$

The geometrical construction of $k_j^{-,+}$ shows that we in fact have $k_0 = \max\{k_{j_1-1}^+, k_{j_2+1}^-\}$, since the integers $k_j^{-,+}$ cannot be greater than the integers between $k_{j_1-1}^+$ and $k_{j_2+1}^-$ in (3.16). The table below (see also Figure 8) presents the full partition of $\mathcal{A}$ into

$$K = 2(j_2 - j_1 + 2)(k_0 + 1) + 2, \tag{3.17}$$

adjacent intervals I_p, which are now indexed from $p = 1$ to $p = K$:

$$\begin{array}{lll}
 & p & I_p \\
 & & \\
L_{j_1-1}: & 1 & [F_-(j_1-1),\, j_1-1] \\
 & 2 & [j_1-1,\, F_{j_1}^{-1}(j_1-1)] \\
 & 3 & [F_{j_1}^{-1}(j_1-1),\, F_{j_1}^{-2}(j_1-1)] \\
 & \dots & \dots \\
 & k_0+2 & [F_{j_1}^{-k_0}(j_1-1), z_{j_1}] \\
\dots & \dots\ \dots & \dots\ \dots \\
L_j: & 2(j-j_1)(k_0+1)+k_0+3 & [z_j,\, F_j^{-k_0}(j)] \\
j \in \{j_1, \dots, j_2\} & \dots & \dots \\
 & 2(j-j_1)(k_0+1)+2k_0+3 & [F_j^{-1}(j),\, j] \\
 & 2(j-j_1+1)(k_0+1)+2 & [j,\, F_{j+1}^{-1}(j)] \\
 & \dots & \dots \\
 & 2(j-j_1+1)(k_0+1)+k_0+2 & [F_{j+1}^{-k_0}(j), z_{j+1}] \\
\dots & \dots\ \dots & \dots\ \dots \\
L_{j_2+1}: & 2(j_2-j_1+1)(k_0+1)+k_0+3 & [z_{j_2+1},\, F_{j_2+1}^{-k_0}(j_2+1)] \\
 & \dots & \dots \\
 & 2(j_2-j_1+1)(k_0+1)+2k_0+3 & [F_{j_2+1}^{-1}(j_2+1),\, j_2+1] \\
 & 2(j_2-j_1+2)(k_0+1)+2 = K & [j_2+1,\, F_+(j_2+1)].
\end{array} \tag{3.18}$$

This partition clearly depends on the parameters a and b through the integers j_1 and j_2, the fixed points z_j, and the functions F_-, F_+, and F_j^{-k}.

Proposition 3.4. *Let us fix $2 \le j_1 \le j_2$ positive integers. Then for any $(a, b) \in P_{j_1}^{j_2}$, the following hold:*

(i) *We have*

$(i1)$ $F(I_1) \supset I_{k_0+3}$ *provided*

$$b \ge g_{j_1}^-(a) = \frac{a^2(a-1)(j_1-1)}{a^2(j_1-1)-a+1}; \tag{3.19}$$

$(i2)$ $F(I_p) = I_{p-1}$ *for any* $p \in \{2, \dots, k_0+1\}$*;*

$(i3)$ $F(I_{k_0+2}) = I_{k_0+1} \cup I_{k_0+2}$.

(ii) *For any integer* $j \in \{j_1, \dots, j_2\}$ *we have*

$(ii1)$ $F(I_p) = I_p \cup I_{p+1}$ *for* $p = 2(j-j_1)(k_0+1)+k_0+3$*;*

$(ii2)$ $F(I_p) = I_{p+1}$ *for any*
$p \in \{2(j-j_1)(k_0+1)+k_0+4, 2(j-j_1)(k_0+1)+2k_0+2\}$*;*

$(ii3)$ $F(I_p) \supset \cup_{q=p+1}^{p+k_0+2} I_q$ *for* $p = 2(j-j_1)(k_0+1)+2k_0+3$*;*

$(ii4)$ $F(I_p) \supset \cup_{q=p-k_0-2}^{p-1} I_q$ *for* $p = 2(j-j_1+1)(k_0+1)+2$*;*

$(ii5)$ $F(I_p) = I_{p-1}$ *for any*
$p \in \{2(j-j_1+1)(k_0+1)+3, 2(j-j_1+1)(k_0+1)+k_0+1\}$*;*

$(ii6)$ $F(I_p) = I_{p-1} \cup I_p$ *for* $p = 2(j-j_1+1)(k_0+1)+k_0+2$.

(iii) *We have*
$(iii1)$ $F(I_p) = I_p \cup I_{p+1}$ *for* $p = 2(j_2 - j_1 + 1)(k_0 + 1) + k_0 + 3$*;*
$(iii2)$ $F(I_p) = I_{p+1}$ *for any*
$\underline{p \in \{2(j_2} - j_1 + 1)(k_0 + 1) + k_0 + 4, 2(j_2 - j_1 + 2)(k_0 + 1)\}$;
$(iii3)$ $\overline{F(I_{K-1})} \supset I_K$ *where* $K - 1 = 2(j_2 - j_1 + 2)(k_0 + 1) + 1$*;*
$(iii4)$ $F(I_K) \supset F(I_{K-k_0-2})$ *provided*

$$b \le g^+_{j_2}(a) = \frac{a^2(a-1)(j_2+1)}{a^2 j_2 + a - 1}. \tag{3.20}$$

Proof. We sketch the calculations for each statement of the proposition in order.
(i1) Since

$$\begin{aligned} F(I_1) &\supset [F(F_-(j_1 - 1)), F_+(j_1 - 1)), \\ I_{k_0+3} &= \left[z_{j_1}, F^{-k_0}_{j_1}(j_1)\right], \end{aligned} \tag{3.21}$$

for statement (i1) to hold we first require

$$F_{j_1-1}((a-b)(j_1-1)) \le z_{j_1} \quad \Leftrightarrow$$

$$a(a-b)(j_1-1) - (j_1-2)b \le \frac{b}{a-1}(j_1-1) \quad \Leftrightarrow \quad b \ge g^-_{j_1}(a).$$

The second requirement

$$F^{-k_0}_{j_1}(j_1) < F_+(j_1 - 1),$$

arising from (3.21), is always fulfilled because of the construction of the integer k_0 (see (3.16)).
(i2) For $p = 2$,

$$F(I_2) = \left[F(j_1-1), F(F^{-1}_{j_1}(j_1-1))\right] = \left[F_-(j_1-1), F^{-0}_{j_1}(j_1-1)\right] = I_1,$$

while for any $p \in \{3, \ldots, k_0 + 1\}$,

$$\begin{aligned} F(I_p) &= \left[F(F^{-(p-2)}_{j_1}(j_1)), F(F^{-(p-1)}_{j_1}(j_1))\right] \\ &= \left[F^{-(p-3)}_{j_1}(j_1), F^{-(p-2)}_{j_1}(j_1)\right] = I_{p-1}. \end{aligned}$$

(i3)

$$\begin{aligned} F(I_{k_0+2}) &= \left[F(F^{-k_0}_{j_1}(j)), F(z_{j_1})\right] = \left[F^{-(k_0-1)}_{j_1}(j), z_{j_1}\right] \\ &= \left[F^{-(k_0-1)}_{j_1}(j), F^{-k_0}_{j_1}(j)\right] \cup \left[F^{-k_0}_{j_1}(j), z_{j_1}\right] = I_{k_0+1} \cup I_{k_0+2}. \end{aligned}$$

(ii1) The calculation is similar to that of (i3) above.
(ii2) Do as in (i2).

(ii3) For $p = 2(j - j_1)(k_0 + 1) + 2k_0 + 3$,

$$\begin{aligned} F(I_p) &\supset \left[F(F_j^{-1}(j)), F_+(j)\right], \\ \cup_{q=p+1}^{p+k_0+2} I_q &= \left[j, F_{j+1}^{-k_0}(j+1)\right]. \end{aligned} \tag{3.22}$$

On the right-hand sides of these expressions, the lower boundaries of the intervals coincide. For the upper boundaries we require

$$F_{j+1}^{-k_0}(j+1) < F_+(j),$$

which is always true as it immediately follows from the definitions of k_0 and k_{j+1}^- (see (3.15) and (3.16)).
Note that (ii1–ii3) reduces to the single formula

$$F(I_p) \supset I_p \cup I_{p+1} \cup I_{p+2} \qquad \text{for} \qquad p = 2(j - j_1) + 3 \qquad \text{if} \qquad k_0 = 0.$$

(ii4) The calculation is the same as above with reference to the definition of the integer k_{j-1}^+.
(ii5) Repeat the above calculation in (i2).
(ii6, iii1) As in (i3).
(iii2) As in (i2).
(iii3) The explanation for the special handling of the interval I_{K-1} is the fact that $F(I_{K-1}) \neq \overline{F(I_{K-1})}$, since F is upper semicontinuous; that is, $F(j_2+1) \neq F_+(j_2+1)$. Thus,

$$\overline{F(I_{K-1})} \supset \left[F(F_{j_2+1}^{-1}(j_2+1)), F_+(j_2+1)\right] = I_K.$$

Note that (iii1)–(iii3) appears in the form

$$\overline{F(I_{K-1})} \supset I_{K-1} \cup I_K \qquad \text{if} \qquad k_0 = 0.$$

(iii4) We have

$$\begin{aligned} F(I_K) &= [F_-(j_2+1), F(F_+(j_2+1))], \\ I_{K-k_0-2} &= \left[F_{j_2+1}^{-k_0}(j_2), z_{j_2+1}\right]; \end{aligned} \tag{3.23}$$

thus for statement (iii4) we need to satisfy

$$F_{j_2+1}^{-k_0}(j_2) > F_-(j_2+1),$$

which is guaranteed by the definition of k_0 and $k_{j_2}^+$. The condition for the upper boundaries in (3.23) yields

$$z_{j_2+1} \le F_{j_2+2}((a-b)(j_2+1)+b) \quad \Leftrightarrow$$

$$\frac{b}{a-1} j_2 \le a(a-b)(j_2+1) + ab - (j_2+1)b \quad \Leftrightarrow \quad b \le g_{j_2}^+(a).$$

This completes the proof. □

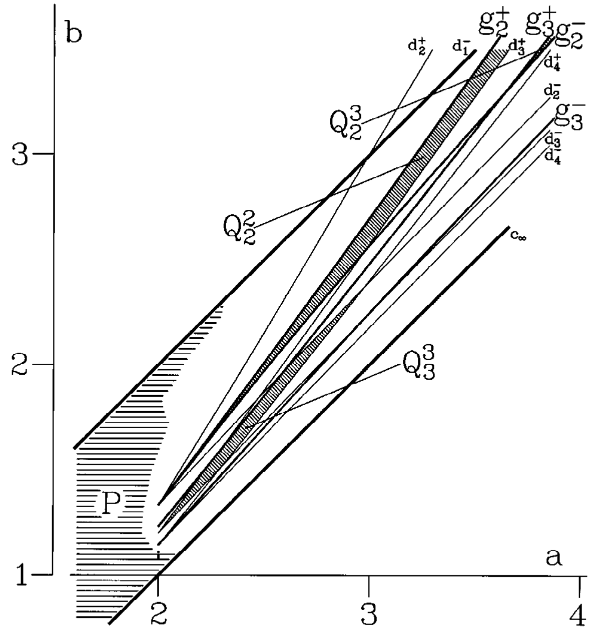

Fig. 9. The parameter domains $Q_{j_1}^{j_2}$.

Let us fix the integers $2 \leq j_1 \leq j_2$ and define the set

$$G_{j_1}^{j_2} = \left\{(a, b) \in P \mid g_{j_1}^-(a) \leq b \leq g_{j_2}^+(a)\right\}.$$

The analysis of the conditions (3.19) and (3.20) by solving $g_{j_1}^-(a) = g_{j_2}^+(a)$ shows that the set $G_{j_1}^{j_2}$ is nonempty for all real values of a if $j_2 < 3j_1 - 4$, and lies in the open half-plane given by

$$a > \frac{1}{2(j_1 - 1)}\left(j_1 + j_2 + \sqrt{j_2^2 + 2j_2(2 - j_1) - (3j_1^2 - 4j_1)}\right).$$

The graph $g_{j_2}^+$ is asymptotic from below to the graph c_{j_2+1}, while the graph $g_{j_1}^-$ is asymptotic from above to the line c_∞. We will use the parameter domain

$$\begin{aligned} Q_{j_1}^{j_2} &= P_{j_1}^{j_2} \cap G_{j_1}^{j_2} \qquad (3.24) \\ &= \left\{(a, b) \in P \mid b \in (d_{j_1}^-(a), d_{j_1-1}^-(a)] \cap [d_{j_2+1}^+(a), d_{j_2}^+(a)) \cap [g_{j_1}^-(a), g_{j_2}^+(a)]\right\}. \end{aligned}$$

Some of these domains and some curves $g_j^{+,-}$ are shown in Figure 9. Note that

$$\begin{aligned} a &< (>) j_1 && \Rightarrow \quad d_{j_1}^-(a) > (<) g_{j_1}^-(a), \\ a &< (>) \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + j_2} && \Rightarrow \quad d_{j_2}^+(a) < (>)\, g_{j_2}^+(a). \end{aligned}$$

The domains $Q_2^{j_2}$ are substantially smaller than their superset $P_2^{j_2}$, but the difference is less and less with increasing values of j_1 (compare, e.g., P_3^3 and Q_3^3).

We now define the matrix $A \in \mathbb{R}^{K\times K}$ by letting

$$a_{ij} = \begin{cases} 1 & \text{if } \overline{F(I_i)} \supset I_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

The following result is an immediate consequence of Proposition 3.4.

Corollary 3.5. *Let us fix the parameters a and b in the definition of the map F and assume that for some integers $2 \le j_1 \le j_2$ $(a, b) \in Q_{j_1}^{j_2}$ holds. Then at least the following elements of the matrix A are nonzero:*

$$\begin{aligned}
&a_{1,k_0+3} = 1; \quad a_{K,K-k_0-2} = 1;\\
&a_{pp} = 1,\\
&p \in \{2(j-j_1+1)(k_0+1)+k_0+2,\, 2(j-j_1+1)(k_0+1)+k_0+3\},\\
&j \in \{j_1-1, \dots, j_2\};\\
&a_{p,p+1} = 1,\\
&p \in \{2(j-j_1)(k_0+1)+k_0+3,\, 2(j-j_1)(k_0+1)+2k_0+3\},\\
&j \in \{j_1, \dots, j_2+1\};\\
&a_{p,p-1} = 1,\\
&p \in \{2(j-j_1+1)(k_0+1)+2,\, 2(j-j_1+1)(k_0+1)+k_0+2\},\\
&j \in \{j_1-1, \dots, j_2\};\\
&a_{p,p+2} = \cdots = a_{p,p+k_0+2} = 1,\\
&p = 2(j-j_1)(k_0+1)+2k_0+3, \qquad j \in \{j_1, \dots, j_2\};\\
&a_{p,p-k_0-2} = \cdots = a_{p,p-2} = 1,\\
&p = 2(j-j_1+1)(k_0+1)+2, \qquad j \in \{j_1, \dots, j_2\}.
\end{aligned} \tag{3.26}$$

As an example, let us consider the case of $j_2 - j_1 = 1$ and $k_0 = 2$, i.e., when $K = 20$. Then the structure of the matrix A will be the following:

$$\left(\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0\\
0 & \cdot & \cdot & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdot & \cdot & 0\\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

Here $\cdot$ denotes elements which may be either 0 or 1 depending on the actual values of the parameters a and b.

The following result is fundamental in our upcoming description of the symbolic dynamics in $\mathcal{A}$.

Proposition 3.6. *Let us fix the parameters a and b in the definition of the map F and assume that for some integers $2 \leq j_1 \leq j_2$ $(a, b) \in Q_{j_1}^{j_2}$ holds. Then all the entries of A^{K-1} are nonzero.*

Proof. The (p, q)-th entry of A^n can be written as

$$a_{pq}^n = \sum_{i_1,\dots,i_{n-1}=1}^{K} a_{pi_1} a_{i_1 i_2} \cdots a_{i_{n-2} i_{n-1}} a_{i_{n-1} q}. \tag{3.27}$$

Since all the entries of A are nonnegative, (3.27) is nonzero if there exists at least one nonzero product in the sum. In the following, we trace in five steps as the nonzero elements in A^n wander with increasing n and as they finally spread into all the entries of A^{K-1}. We will use the nonzero elements of A presented in Corollary 2.5 without explicit reference to those formulas, and also make use of the fact that $a_{pp}^n \neq 0$ whenever $a_{pp} = 1$.

(1) The entries of A^{k_0+1} are nonzero at the following indices:

(i) At $(k_0 + 2, 1), (k_0 + 2, 2), \dots, (k_0 + 2, k_0 + 2)$, since

$$a_{k_0+2,k_0+2}^{n-1} a_{k_0+2,k_0+1} a_{k_0+1,k_0} \cdots a_{n+1,n} \neq 0 \quad \Rightarrow$$

$$a_{k_0+2,n}^{k_0+1} \neq 0 \qquad \text{for any } n \in \{1, \dots, k_0\}.$$

(ii) Similarly, at $(K - k_0 - 1, K - k_0 - 1), (K - k_0 - 1, K - k_0), \dots, (K - k_0 - 1, K)$.

(iii) At $(p, p), (p, p + 1), \dots, (p, p + 2k_0 + 2)$ for $p = 2(j - j_1)(k_0 + 1) + k_0 + 3,\ j \in \{j_1, \dots, j_2\}$, since

$$a_{pp}^{k_0-n-1} a_{p,p+1} a_{p+1,p+2} \cdots a_{p+n-1,p+n} \neq 0 \quad \Rightarrow$$

$$a_{p,p+n}^{k_0+1} \neq 0 \qquad \text{for any } n \in \{0, \dots, k_0 + 1\},$$

and

$$a_{p,p+1} a_{p+1,p+2} \cdots a_{p+k_0-1,p+k_0} a_{p+k_0,p+n} \neq 0 \quad \Rightarrow$$

$$a_{p,p+n}^{k_0+1} \neq 0 \qquad \text{for any } n \in \{k_0 + 2, \dots, 2k_0 + 2\}.$$

(iv) Similarly, at $(p, p - 2k_0 - 2), (p, p - 2k_0 - 1), \dots, (p, p)$ for $p = 2(j - j_1 + 1)(k_0 + 1) + k_0 + 2,\ j \in \{j_1, \dots, j_2\}$.

(2) The entries of $A^{(j_2-j_1+2)(k_0+1)}$ are nonzero at the following indices:

(i) At $(k_0 + 3, k_0 + 3), (k_0 + 3, k_0 + 4), \dots, (k_0 + 3, K)$ since

$$a_{k_0+3,k_0+3}^{j_2-j+1} a_{k_0+3,3k_0+5}^{k_0+1} a_{3k_0+5,5k_0+7}^{k_0+1} \cdots$$

$$\cdots a_{2(j-j_1)(k_0+1)+k_0+3,2(j-j_1)(k_0+1)+k_0+3+n}^{k_0+1} \neq 0 \quad \Rightarrow$$

$$a^{(j_2-j_1+1)(k_0+1)}_{k_0+3,2(j-j_1)(k_0+1)+k_0+3+n} \neq 0,$$

for any $n \in \{0, \dots, 2(k_0+1)\}$, $j \in \{j_1, \dots, j_2\}$, and
for any $n \in \{0, \dots, k_0+2\}$, $j = j_2+1$.

(ii) Similarly, at $(K-k_0-2, 1), (K-k_0-2, 2), \dots, (K-k_0-2, K-k_0-2)$.

(3) The entries of $A^{(j_2-j_1+2)(k_0+1)+1}$ are nonzero at the following indices:

(i) At $(1, k_0+3), (1, k_0+4), \dots, (1, K)$ since

$$a_{1,k_0+3}a^{(j_2-j_1+1)(k_0+1)}_{k_0+3,n} \neq 0 \quad \Rightarrow$$

$$a^{(j_2-j_1+2)(k_0+1)+1}_{1,n} \neq 0 \qquad \text{for any } n \in \{k_0+3, \dots, K\}.$$

(ii) Similarly, at $(K, 1), (K, 2), \dots, (K, K-k_0-2)$.

(4) The entries of $A^{(j_2-j_1+2)(k_0+1)+k_0+2}$ are nonzero at the following indices:

(i) At $(k_0+2, 1), (k_0+2, 2), \dots, (k_0+2, K)$ in the whole k_0+2-nd row, i.e., $a^{(j_2-j_1+2)(k_0+1)+k_0+2}_{k_0+2,q} \neq 0$, $q \in \{1, \dots, K\}$, since

$$a^{k_0+1}_{k_0+2,1}a^{(j_2-j_1+2)(k_0+1)+1}_{1,n} \neq 0 \quad \Rightarrow$$

$$a^{(j_2-j_1+2)(k_0+1)+k_0+2}_{k_0+2,n} \neq 0 \qquad \text{for any } n \in \{k_0+3, \dots, K\},$$

and

$$a^{(j_2-j_1+2)(k_0+1)+1}_{k_0+2,k_0+2}a^{k_0+1}_{k_0+2,n} \neq 0 \quad \Rightarrow$$

$$a^{(j_2-j_1+2)(k_0+1)+k_0+2}_{k_0+2,n} \neq 0 \qquad \text{for any } n \in \{1, \dots, k_0+2\}.$$

(ii) Similarly, at $(K-k_0-1, 1), (K-k_0-1, 2), \dots, (K-k_0-1, K)$, i.e., in the whole $K-k_0-1$-st row.

(5) The entries of $A^{2(j_2-j_1+2)(k_0+1)+1} = A^{K-1}$ are nonzero at the following indices:

(i) At (p, q) for all $p = 2(j-j_1+1)(k_0+1)+2+n$, $n \in \{0, \dots, k_0\}$, $j \in \{j_1, \dots, j_2\}$ and $q \in \{1, \dots, K\}$ since

$$a_{p,p-1}a_{p-1,p-2}\cdots a_{2(j-j_1+1)(k_0+1)+3,2(j-j_1+1)(k_0+1)+2}\cdot$$

$$\cdot a_{2(j-j_1+1)(k_0+1)+2,2(j-j_1)(k_0+1)+k_0+2}\cdots\cdots\cdots$$

$$\cdots\cdots\cdots a_{4k_0+7,4k_0+6}a_{4k_0+6,3k_0+4}a_{3k_0+4,3k_0+3}\cdots$$

$$\cdots a_{2k_0+6,2k_0+5}a_{2k_0+5,2k_0+4}a_{2k_0+4,k_0+2}\cdot$$

$$\cdot a^{(j_2-j+1)(k_0+1)-1-n}_{k_0+2,k_0+2}a^{(j_2-j_1+2)(k_0+1)+k_0+2}_{k_0+2,q} \neq 0 \quad \Rightarrow a^{K-1}_{pq} \neq 0.$$

(ii) Similarly, at (p, q) for all $p = 2(j-j_1)(k_0+1)+k_0+3+n$, $n \in \{0, \dots, k_0\}$, $j \in \{j_1, \dots, j_2\}$ and $q \in \{1, \dots, K\}$.

(iii) At (p, q) for all $p \in \{1, k_0 + 2\}$ and $q \in \{1, \ldots, K\}$, since

$$a_{p,p-1} \cdots a_{32} a_{21} a_{1,k_0+3} a_{k_0+3,2k_0+4}^{k_0+1} a_{2k_0+4,k_0+2} \cdot$$

$$\cdot a_{k_0+2,k_0+2}^{(j_2-j_1+1)(k_0+1)-2-p} a_{k_0+2,q}^{(j_2-j_1+2)(k_0+1)+k_0+2} \neq 0 \quad \Rightarrow a_{pq}^{K-1} \neq 0.$$

(iv) Similarly, at (p, q) for all $p \in \{K - k_0 - 1, K\}$ and $q \in \{1, \ldots, K\}$.

The subcases (i–iv) cover all the possible indices in A^{K-1}; that is, $a_{pq}^{K-1} \neq 0$ for any $p, q \in \{1, \ldots, K\}$.

This completes the proof of the proposition. □

We are now in the position to prove that under the conditions of the previous proposition the invariant set $\mathcal{A}$ cannot be decomposed into smaller invariant sets of nonzero measure.

Proposition 3.7. *Let us suppose that the assumptions of Proposition 3.6 hold. Then F is topologically transitive on $\mathcal{A}$.*

Proof. We have to show that if $U, V \subset \mathcal{A}$ are disjoint open sets, then there exists $n \geq 1$ such that $F^n(U) \cap V \neq \emptyset$. We will prove this in two steps. First, we show that for any open set $U \subset \mathcal{A}$ there exists an integer $m \geq 1$ such that $F^m(U)$ contains a full interval I_p of the partition (3.18) for some $1 \leq p \leq K$. Let us suppose the contrary, i.e., U is a connected open interval in $\mathcal{A}$ such that its image under F^m does not contain a full interval I_p for arbitrary $m \geq 1$. Let $l(W)$ denote the length of an open connected interval W and let

$$L = l(U).$$

Now U must have a connected open subset $U_1 \subset U$ such that F is continuous on U_1. Moreover, by our assumption on U, U_1 can be chosen such that

$$l(U_1) > \frac{L}{2}.$$

Then $F(U_1)$ is a connected open set with

$$l\left(F(U_1)\right) > \frac{La}{2},$$

and it has a connected open subset $U_2 \subset F(U_1)$ on which F is continuous and

$$l(U_2) > \frac{La}{4}.$$

This again implies that

$$l(F(U_2)) > \frac{La^2}{4}.$$

Moreover, by assumption, $F(U_2)$ does not contain a full interval I_p; hence it has a connected open subset $U_3 \subset F(U_2)$ on which F is continuous and

$$l(U_3) > \frac{La^2}{8}.$$

Repeating this construction, we can build a sequence $\{U_m\}_{m=1}^{\infty}$ of connected open sets, such that none of these contains a full interval I_p and hence their lengths satisfy

$$l(U_m) < 1. \tag{3.28}$$

On the other hand, as above, we have the estimate

$$l(U_m) > \frac{1}{2}L\left(\frac{a}{2}\right)^{m-1}. \tag{3.29}$$

Since $L \neq 0$ and $(a, b) \in P_{j_1}^{j_2}$ implies $a > 2$, (3.29) contradicts (3.28) for m sufficiently large, which completes the first part of the proof.

Using this result we may now assume that for some $m \geq 1$, F^m contains a full interval I_p for some $1 \leq p \leq K$. Since $V \subset \mathcal{A}$ is open, there exists $q \in \{1, \ldots, K\}$ such that $(I_q - \partial I_q) \cap V \neq \emptyset$. By Proposition 3.6, we have

$$a_{pq}^{K-1} \neq 0;$$

hence we can find an index sequence $i_1, \ldots, i_{K-2}$ such that

$$a_{pi_1} a_{i_1 i_2} \cdots a_{i_{K-2},q} \neq 0.$$

By (3.25) this means that

$$\begin{aligned}
\overline{F(I_p)} &\supset I_{i_1}, \\
\overline{F^2(I_p)} &\supset \overline{F(I_{i_1})} \supset I_{i_2}, \\
&\vdots \\
\overline{F^{K-1}(I_p)} &\supset \overline{F^{K-2}(I_{i_1})} \supset \overline{F^{K-3}(I_{i_2})} \supset \cdots \supset I_q,
\end{aligned}$$

implying

$$F^{K-1+m}(U) \cap V \supset F^{K-1}(I_p) \cap V \supset (I_q - \partial I_q) \cap V \neq \emptyset,$$

which concludes the proof. $\square$

We now summarize the results of this section in the following theorem.

Theorem 3.8. *Let us fix the parameters a and b in the definition of F and assume that for some integers $2 \leq j_1 \leq j_2$, $(a, b) \in Q_{j_1}^{j_2}$ holds. Then the set $\mathcal{A} \subset I$ defined in (3.9) is a hyperbolic strange attractor for the map F.*

Proof. By Propositions 3.3 and 3.7, $\mathcal{A}$ is a closed, indecomposable attracting set, and hence it is an attractor. By Propositions 3.1 and 3.7, F is a chaotic map restricted to $\mathcal{A}$, and hence $\mathcal{A}$ is a strange attractor. Finally, $F'(x) = a > 1$ for any $x \in \mathcal{A}$ (the upper derivative of F always exists), and hence $\mathcal{A}$ is hyperbolic. $\square$

The above theorem proves that the set $\mathcal{A}$ is a strange attractor in the parameter domains $Q_{j_1}^{j_2}$. One of the main ingredients used in the proof is the topological transitivity of the map F on $\mathcal{A}$. We established this property of F in Proposition 3.7 using symbolic dynamics based on a partition of $\mathcal{A}$. Our construction required conditions (3.20) and (3.19), which restrict the domain of existence of the attractor $\mathcal{A}$ in the parameter plane. It appears, however, that more involved interval decompositions of $\mathcal{A}$ would yield the topological transitivity of F on larger parameter domains that would ultimately cover the whole domain $P_{j_1}^{j_2}$ with the exception of a measure zero set. We also remark that the simplest possible partition of $\mathcal{A}$ corresponds to $k_0 = 0$, in which case the domain of existence of $\mathcal{A}$ is substantially smaller than what we obtained for more sophisticated partitions with $k_0 > 0$.

4. Symbolic Dynamics on $\mathcal{A}$

In this section we will examine the dynamics within the attractor $\mathcal{A}$ more closely. As the reader can expect, the partition of the last section can be used to introduce a symbolic characterization of $\mathcal{A}$. In fact, $\{I_j\}_{j=1}^K$ is a *Markov partition* of $\mathcal{A}$ (see, e.g., [2]). Since our map is not even continuous over this partition, we provide here more details than usual for the construction which is standard for Markov transformations (our map F is not Markov in the sense of, e.g., [3] or [4]).

As a first step, we slightly modify the partition of $\mathcal{A}$ used in the previous section. Namely, we let

$$\begin{aligned} \hat{I}_p &= I_p, \qquad\qquad p = 1, \ldots, K-1, \\ \hat{I}_K &= [j_2 + 1, F_+(j_2+1) - \varepsilon], \end{aligned} \tag{4.1}$$

where K and I_p are defined in (3.17) and (3.18), respectively, and $\varepsilon \geq 0$ is a yet undetermined small parameter. Note that for $\varepsilon > 0$ this partition does not cover all of the attractor $\mathcal{A}$. We also change the definition of the matrix A by defining $\hat{A} \in \mathbb{R}^{K\times K}$ as

$$\hat{a}_{ij} = \begin{cases} 1 & \text{if } F(I_i) \supset I_j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Notice that we now require $F(I_i)$ (as opposed to its closure in (3.25) to contain the interval I_j for the case $\hat{a}_{ij} = 1$). The reason for these changes is that for our construction we will need the preimage of any I_i under F to be a *closed* subset of I_j whenever $\hat{a}_{ij} \neq 0$. Of course, we would like to retain the nice properties of A for our modified set-up, which can be achieved by the proper choice of the parameter ε in (4.1).

Proposition 4.1. *Let us fix the parameters a and b in the definition of the map F and* *assume that for some integers* $2 \leq j_1 \leq j_2$, $(a, b) \in \hat{Q}_{j_1}^{j_2}$ *holds with*

$$\hat{Q}_{j_1}^{j_2} = \left\{ (a, b) \in P \mid \in (d_{j_1}^-(a), d_{j_1-1}^-(a)] \cap [d_{j_2+1}^+(a), d_{j_2}^+(a)) \cap [g_{j_1}^-(a), g_{j_2}^+(a)) \right\}. \tag{4.3}$$

Then we can select $\varepsilon > 0$ *small in (4.1) such that* $A = \hat{A}$. *In particular, all the entries of* $\hat{A}^{K-1}$ *are nonzero.*

Proof. A quick review of the statements of Proposition 3.4 shows that for $\varepsilon = 0$ in (4.1) all the entries of A and $\hat{A}$ are the same except for

$$1 = a_{K-1,K} \neq \hat{a}_{K-1,K}|_{\varepsilon=0} = 0.$$

However, one can select some appropriately small

$$0 < \varepsilon < F_+(j_2 + 1) - F_{j_2+2}^{-1}(z_{j_2+1})$$

such that

$$F(\hat{I}_{K-1}) \supset \hat{I}_K$$

is satisfied and

$$F(\hat{I}_K) \supset \hat{I}_{K-k_0-2}$$

still holds, which implies

$$\hat{a}_{K-1,K}|_{\varepsilon>0} = 1,$$

with all the other entries of $\hat{A}$ equal to the respective entries of A. □

Note the slight restriction on the parameter set by using $\hat{Q}_{j_1}^{j_2} \subset Q_{j_1}^{j_2}$ in (4.3) instead of (3.24). This is needed for the construction of ε in the above proposition.

In view of Proposition 4.1, we now fix (a, b), *select an appropriate* $\varepsilon > 0$, *and drop the hats from* $\hat{A}$, $\hat{I}_K$, *and* $\hat{Q}_{j_1}^{j_2}$, *keeping their new definitions in mind.*

We start the description of the dynamics on the set $\mathcal{A}$ by introducing the set

$$\mathcal{K} = \{1, 2, \ldots, K\},$$

and considering the set of semi-infinite symbol sequences

$$\Sigma_A = \{s = s_0 s_1 \cdots s_i \cdots \mid s_i \in \mathcal{K},\ a_{s_i s_{i+1}} = 1, i \in \mathbb{Z}^+\}.$$

As is well known (see, e.g., [18]), Σ_A is a complete metric space with the metric

$$d(s, \bar{s}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|s_i - \bar{s}_i|}{1 + |s_i - \bar{s}_i|}. \tag{4.4}$$

This metric has the following two properties:

(1) If $s_i = \bar{s}_i$ for $i = 0, \ldots, N$ then $d(s, \bar{s}) < 1/2^N$.
(2) If $d(s, \bar{s}) < 1/2^{N+1}$ then $s_i = \bar{s}_i$ for $i = 1, \ldots, N$.

The matrix A is usually called the *transition matrix* corresponding to the space Σ_A. It can be shown that if A is irreducible, then Σ_A is a compact, totally disconnected, and perfect space—in other words, a *Cantor set*. The irreducibility of A means that it has some power with all nonzero elements; hence in our case A is irreducible if the conditions of Proposition 4.1 hold. On Σ_A one can define a map

$$\begin{aligned}\pi_A\colon\ &\Sigma_A \to \Sigma_A,\\ s &= s_0 s_1 s_2 \cdots \mapsto \tilde{s} = s_1 s_2 s_3 \ldots,\end{aligned}$$

which is a *subshift of finite type* with transition matrix A. The map π_A has

(1) a countable infinity of periodic orbits,
(2) an uncountable infinity of nonperiodic orbits,
(3) a dense orbit.

In the following we will show that $\mathcal{A}$ has a subset on which F behaves in much the same way as π_A.

Let

$$\mathcal{E} = \cup_{p=1}^{K} \partial I_p \tag{4.5}$$

be the set containing all the boundary points of the intervals $I_1, \ldots, I_K$ and define the sets

$$\mathcal{L} = \mathcal{A} \cap \cup_{s\in\Sigma_A} \cap_{i=0}^{\infty} F^{-i}(I_{s_i}), \qquad \mathcal{B} = \mathcal{L} \cap \cup_{i=0}^{\infty} F^{-i}(\mathcal{E}), \qquad \Lambda = \mathcal{L} - \mathcal{B}, \tag{4.6}$$

and the map

$$\begin{aligned}&S\colon\ \Sigma_A \to \mathcal{L},\\ &s \mapsto x, \quad \Leftrightarrow \quad F^i(x) \in I_{s_i}, \qquad \text{for all } i = 0, 1, \ldots.\end{aligned} \tag{4.7}$$

We can now prove the following result.

Proposition 4.2. *Suppose that the conditions of Proposition 4.1 hold. Then*

(i) The map $S\colon\ \Sigma_A \to \mathcal{L}$ is well-defined. In fact, it is a continuous surjection.
(ii) $S|S^{-1}(\Lambda)$ is a homeomorphism onto its image.
(iii) $F|\mathcal{L}$ is topologically semiconjugate to π_A, i.e., the following diagram commutes:

$$\begin{array}{ccc}\Sigma_A & \xrightarrow{S} & \mathcal{L}\\ {\scriptstyle\pi_A}\downarrow & & \downarrow{\scriptstyle F}\\ \Sigma_A & \xrightarrow[S]{} & \mathcal{L}\end{array}$$

Proof. To prove (i), we first show that for any $s \in \Sigma_A$ the set

$$G = \cap_{i=0}^{\infty} F^{-i}(I_{s_i})$$

is not empty. Let $s = s_0 s_1 \cdots \in \Sigma_A$. By the definition of Σ_A we know that

$$F(I_{s_0}) \supset I_{s_1} \quad \Leftrightarrow \quad G_1 = F^{-1}(I_{s_1}) \cap I_{s_0} \neq \emptyset.$$

Furthermore, G_1 is a closed subset of I_{s_0} (even if I_{s_0} contains a discontinuity of F, as one can easily see). We also know from the definition of Σ_A that

$$F(I_{s_1}) \supset I_{s_2} \quad \Leftrightarrow \quad F^{-1}(I_{s_2}) \cap I_{s_1} \neq \emptyset,$$

and hence

$$G_2 = F^{-1}(F^{-1}(I_{s_2}) \cap I_{s_1}) \cap I_{s_0} \subset G_1$$

is a nonempty closed set. In general,

$$G_j = F^{-1}(\ldots(F^{-1}(F^{-1}(I_{s_j}) \cap I_{s_{j-1}}) \cap \ldots) \cap I_{s_1}) \cap I_{s_0} = \cap_{i=0}^{j} F^{-i}(I_{s_i}) \subset G_{j-1}$$

is a nonempty closed set. Then

$$G = \cdots \subset G_j \subset G_{j-1} \subset \cdots \subset G_1 \subset G_0 \equiv I_{s_0}, \tag{4.8}$$

as the intersection of a nested sequence of closed nonempty intervals, is nonempty.

This result implies that for any $s \in \Sigma_A$ there exists $x \in \mathcal{L}$ such that $F^i(x) \in I_{s_i}$ for all $i \in \mathbb{Z}^+$. If this x is unique, then the map S is well-defined. Suppose the contrary, i.e., there exists $y \neq x$ in $\mathcal{L}$ with $F^i(y) \in I_{s_i}$ for all $i \in \mathbb{Z}^+$. Then no iterate of the open interval $U = (x, y)$ under F contains a full interval I_k which contradicts our first observation in the proof of Proposition 3.7.

The fact that S is onto follows immediately from the definition of $\mathcal{L}$, since if $x \in \mathcal{L}$ then there exists $s \in \Sigma_A$ such that

$$x \in \cap_{i=0}^{\infty} F^{-i}(I_{s_i}) \quad \Leftrightarrow \quad S(s) = x.$$

We now show that S is continuous. Let us fix $\varepsilon > 0$ and let

$$\delta(\varepsilon) = \frac{1}{2^{N(\varepsilon)+1}}, \qquad N(\varepsilon) = \mathrm{Int}\left(\log_a \frac{1}{\varepsilon}\right) + 1.$$

By property (2) of the metric d in (4.4), if $\bar{s} \in \Sigma_A$, then

$$d(s, \bar{s}) < \delta \quad \Rightarrow \quad s_i = \bar{s}_i, \qquad i = 1, \ldots, N(\varepsilon),$$

implying

$$S(s), S(\bar{s}) \in \cap_{i=0}^{N(\varepsilon)} F^{-i}(I_{s_i}).$$

Since the length $l(I_k)$ of any interval I_k is less than 1, we have

$$|S(s) - S(\bar{s})| < \frac{1}{a^{N(\varepsilon)}} l(I_{s_{N(\varepsilon)}}) < \frac{1}{a^{N(\varepsilon)}} < \varepsilon,$$

and hence S is continuous, which proves statement (i).

Clearly, S is one-to-one over the set $S^{-1}(\Lambda)$, and thus S^{-1} is well-defined on this set. Indeed, since F is well-defined, for any $x \in \Lambda$ there exists a unique sequence $s_0, s_1, \ldots$ such that $F^i(x) \in I_{s_i}$, $i \in \mathbb{Z}^+$. Consequently, to prove (ii) we only have to show that S^{-1} is continuous. Since F^j is continuous on Λ, for any $x \in \Lambda$ and $n \geq 1$, there exists $\delta(n) > 0$ such that if $y \in \Lambda$, then

$$|x - y| < \delta(n) \quad \Rightarrow \quad F^j(x), F^j(y) \in I_{s_j}, \qquad j = 1, \ldots, n,$$

with $s = S^{-1}(x)$. Fix $\varepsilon > 0$ and select

$$n = \operatorname{Int}\left(\log_2 \frac{1}{\varepsilon}\right) + 1.$$

If $\bar{s} = S^{-1}(y)$, then by property (1) of the metric d defined in (4.4) we have

$$d(s, \bar{s}) < \varepsilon,$$

and hence S^{-1} is continuous. This proves statement (ii).

Finally, consider $s = s_0 s_1 \cdots \in \Sigma_A$. Then, by the definition of π_A, $\pi_A(s) = s_1 s_2 \cdots$ and

$$S \circ \pi_A(s) = \cap_{i=1}^{\infty} F^{-i}(I_{s_i}).$$

On the other hand,

$$S(s) = \cap_{i=0}^{\infty} F^{-i}(I_{s_i}) \quad \Rightarrow \quad F \circ S(s) = \cap_{i=1}^{\infty} F^{-i}(I_{s_i}),$$

which proves that $(F|\mathcal{L}) \circ S = S \circ \pi_A$. We can therefore conclude that the semiconjugacy claimed in statement (iii) of the proposition indeed holds. $\square$

We can summarize the main results of this section together with their consequences as follows.

Theorem 4.3. *Let us suppose that the conditions of Proposition 4.1 hold. Then*

(i) $\Lambda \subset \mathcal{A}$ *is a hyperbolic invariant Cantor set for the map* F*, on which* F *is topologically semiconjugate to a one-sided subshift on* K *symbols with the irreducible transition matrix* A*.*

(ii) Let $n \geq 2$ *be an integer. Then on* Λ F *has*

$$N(n) = \frac{1}{n}\left(\operatorname{Tr} A^n - \sum_{\langle i,n\rangle} i N(i)\right) \tag{4.9}$$

distinct periodic orbits of minimal period n*, where the notation* $\langle i, n\rangle$ *refers to integers* $1 \leq i < n$ *which divide* n*.*

(iii) F *has an uncountable infinity of nonperiodic orbits and an orbit which is dense in* Λ*.*

Proof. Statement (i) follows directly from Propositions 4.2 and the properties of A listed in (3.26) in accordance with Proposition 3.6. Statement (iii) follows from the properties of Σ_A and π_A (see, e.g., [18]). By the topological semi-conjugacy of F to π_A, it is enough to show that (4.9) in (ii) holds for π_A.

Let us count the number of distinct periodic orbits of period n for π_A. Any of these periodic orbits is associated with a periodic sequence $s_0 s_1 \cdots s_n s_0 s_1 \cdots s_n \cdots$. The number of n-long symbol sequences beginning with p and ending with q, which can appear within an element s of Σ_A, is exactly the number of nonzero terms in the sum

$$\sum_{i_1,\dots,i_{n-1}=1}^{K} a_{pi_1} a_{i_1 i_2} \cdots a_{i_{n-2} i_{n-1}} a_{i_{n-1} q} = a_{pq}^n.$$

Therefore, the number of possible n-long periodic symbol sequences is given by

$$P(n) = \sum_{p=1}^{K} a_{pp}^n = \operatorname{Tr} A^n.$$

Some of these orbits, however, have prime periods lower than n. The number of these are subtracted from $P(n)$ in (4.9). Furthermore, if n infinite symbol sequences form an n-periodic orbit for π_A, the above argument counts that single orbit n times, depending on which of the n symbol sequences we start iterating π_A at. Based on these observations, the formula in (ii) follows. $\square$

We finish this section with a statement on the domain of attraction of the chaotic attractor $\mathcal{A}$. In the previous section we saw that it always contains the open set $\mathcal{D}_0$ defined in (3.12).

Theorem 4.4. *Let us suppose that the conditions of Proposition 4.1 are satisfied and let $\mathcal{D}$ denote the domain of attraction of $\mathcal{A}$. Moreover,*

(i) suppose that $j_1 = 2$ and $j_2 = N - 1$ with N defined in (2.6), i.e., F has no fixed points outside $\mathcal{A} \cup \{0\}$. Then $\mathcal{D} = I - \{0\}$.

(ii) Suppose that there exists j with $j_2 < j \le N - 1$ such that

$$b \le h_j^+(a) = \frac{j(a^2-1)}{j(a+1)-a}$$

holds. Then the interval

$$[z_j, F_+(j)] = \left[\frac{b(j-1)}{a-1}, (a-b)j + b\right]$$

contains an invariant Cantor set $\mathcal{C}_j \not\subset \mathcal{D}$. On $\mathcal{C}_j$ F is topologically conjugate to a one-sided subshift on two symbols with transition matrix

$$A_j = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iii) Suppose that there exists j *with* $1 \le j < j_1$ *such that*

$$b \ge h_j^-(a) = \frac{j(a^2-1)}{j(a+1)-1}$$

holds. Then the interval

$$[F_-(j), z_{j+1}] = \left[(a-b)j, \frac{bj}{a-1}\right]$$

contains an invariant Cantor set $\mathcal{C}_j \not\subset \mathcal{D}$ *with properties as in (ii) above.*

Proof. Statement (i) follows from the proof of Proposition 3.3, so we start with the proof of (ii). Let us define the intervals

$$J_1 = [z_j, j] = \left[\frac{b(j-1)}{a-1}, j\right],$$

$$J_2 = [F_{j+1}^{-1}, F_+(j)] = \left[\frac{b(aj-1)}{a(a-1)}, (a-b)j+b\right].$$

We then have

$$\overline{F(J_1)} = [z_j, F_+(j)] \supset J_1 \cup J_2. \tag{4.10}$$

Furthermore, we have

$$F(J_2) = [F_{j+1}(F_{j+1}^{-1}(z_j)), F_{j+1}(F_+(j))] = \left[\frac{b(j-1)}{a-1}, a\left((a-b)j+b\right) - jb\right],$$

which implies

$$F(J_2) \supset J_1 \quad \Leftrightarrow \quad a\left((a-b)j+b\right) - jb \ge j \quad \Leftrightarrow \quad b \le h_j^+(a). \tag{4.11}$$

Then using the same methods as in Proposition 4.2, one obtains statement (ii) from (4.10) and (4.11).

Similarly, to prove (iii) we let

$$J_2 = [F_-(j), F_j^{-1}(z_{j+1})] = \left[(a-b)j, \frac{b\left(a(j-1)+1\right)}{a(a-1)}\right],$$

$$J_1 = [j, z_{j+1}] = \left[j, \frac{bj}{a-1}\right].$$

We then have

$$F(J_1) = [F_-(j), z_{j+1}] \supset J_1 \cup J_2, \tag{4.12}$$

and

$$F(J_2) = [F_j(F_-(j)), F_j(F_j^{-1}(z_{j+1}))] = \left[a(a-b)j - (j-1)b, \frac{bj}{a-1}\right],$$

from which we obtain

$$F(J_2) \supset J_1 \quad \Leftrightarrow \quad a(a-b)j - (j-1)b \le j \quad \Leftrightarrow \quad b \ge h_j^-(a). \tag{4.13}$$

Again, (4.12) and (4.13) imply statement (iii) of the theorem. $\square$

Remark 4.1. Note that the above theorem shows that if F has no fixed points outside $\mathcal{A} \cup \{0\}$, then iterating an arbitrary point $x \in \mathcal{A} \cup \{0\}$, we monotonically approach the attractor $\mathcal{A}$ and become under the influence of the chaotic dynamics inside $\mathcal{A}$. If, however, F has fixed points outside $\mathcal{A} \cup \{0\}$, then there may exist an open set of initial conditions in $\mathcal{D} - \mathcal{D}_0$ such that the corresponding trajectories of F first become under the influence of chaotic invariant sets which are horseshoes for F^2. In practice this means that the trajectories undergo a *transient chaos* before they arrive at the chaotic attractor (see also the iteration shown in Figure 7). This fact shows that even the domain of attraction $\mathcal{D}$ of $\mathcal{A}$ may have a complex structure. In particular, the attractor may have a fractal basin boundary (see, e.g., [14] for examples and details).

5. Some Properties of the Hyperbolic Set $\Lambda \subset \mathcal{A}$

Using the fact that the dynamics of F restricted to Λ is topologically semiconjugate to a subshift of finite type, one can use general results on subshifts to characterize F on this invariant set. For example, the *topological entropy* $h_{\bar{F}}$ of $\bar{F} \equiv F|\Lambda$ (with respect to the Lebesgue-measure on I) can be computed as

$$h_{\bar{F}} = \log |\lambda|_{\max}, \tag{5.1}$$

where $|\lambda|_{\max} > 1$ is the dominant eigenvalue of the transition matrix A (see, e.g., Mane [12]). From this we immediately obtain the following.

Proposition 5.1. *Suppose that the conditions of Corollary 3.5 are satisfied and all the elements of A not listed in (3.26) are zero. Then we have*

$$h_{\bar{F}} \leq \log(k_0 + 3),$$

with k_0 defined in (3.16).

Proof. Note that under the conditions of Corollary 3.5 the sum of off-diagonal elements in any row of A is at most $k_0 + 2$ while the diagonal element is at most 1 in any row. Then the statement of the proposition is an immediate consequence of (5.1) and the Gershghorin theorem (see, e.g., [5]). □

Another characteristic quantity, the Ljapunov exponent of $\bar{F}$, can directly be computed from the definition of F

$$\mu_{\bar{F}} = \mu_F = \log a,$$

noting that the upper derivative of F exists at any point of I.

Although the Lebesgue measure of the set Λ is zero, it still has a decisive effect on the dynamics within $\mathcal{A}$. Using $\mu_{\bar{F}}$ and $h_{\bar{F}}$ we can give an estimate for the "size" of the invariant set Λ.

Theorem 5.2. *Suppose that the conditions of Proposition 4.1 hold. Then the Haussdorff-dimension $HD(\Lambda)$ and the capacity (or fractal dimension) $C(\Lambda)$ of Λ obey the estimates*

$$HD(\Lambda), C(\Lambda) \leq \frac{h_{\bar{F}}}{\mu_{\bar{F}}}.$$

Proof. We first prove the estimate involving the Haussdorff-dimension of the hyperbolic set. By definition, $HD(\Lambda)$ is the infimum of the numbers $\alpha > 0$, such that for any ε there exist $\delta > 0$ and a covering $\{C_i\}_{i=1}^{\infty}$ of Λ by closed intervals (1-balls) such that $l(C_i) < \delta$ and $\sum_i l^{\alpha}(C_i) < \varepsilon$. First we will construct a covering of Λ such that the length of the individual closed intervals will be as small as needed.

Fix a number $j \in \mathcal{K}$ and suppose that $x \in \Lambda \cap I_j$. This implies the existence of a unique $s \in \Sigma_A$ with $s_0 = j$ such that

$$x \in G_n(s_1, \ldots, s_n) = \cap_{i=0}^{n} F^{-i}(I_{s_i}),$$

for any $n \geq 0$ integer. Therefore,

$$x \in \Lambda \cap I_{s_0} \quad \Rightarrow \quad x \in \cup_{a_{s_0,s_1} \cdots a_{s_{n-1},s_n} \neq 0} G_n(s_1, \ldots, s_n), \tag{5.2}$$

and hence for any $n \geq 0$, $\Lambda \cap I_{s_0}$ can be covered by a number of

$$\sum_{i_1,\ldots,i_n=1}^{K} a_{s_0,i_1} \cdots a_{i_{n-1},i_n}$$

intervals, and each has a length less than

$$\delta(n) = \frac{1}{a^n}.$$

Hence, for any $n \geq 0$ we have a number $\delta(n) > 0$ such that Λ can be covered by a number of

$$N(\delta(n)) = \sum_{i_0,\ldots,i_n=1}^{K} a_{s_0,i_1} \cdots a_{i_{n-1},i_n} = \|A^n\|_s \tag{5.3}$$

intervals, each with length less than $\delta(n)$. Note that in (5.3) $\|A\|_s$ denotes the norm obtained by summing all the elements of the nonnegative matrix A. To estimate the Haussdorff-dimension of Λ, we seek the infimum of $\alpha > 0$ such that

$$\lim_{n\to\infty} \|A^n\|_s \left(\frac{1}{a^n}\right)^{\alpha} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \log\left(\|A^n\|_s \left(\frac{1}{a^n}\right)^{\alpha}\right) = -\infty. \tag{5.4}$$

By the equivalence of the norm $\| \, . \, \|_s$ to the Euclidean matrix norm $\| \, . \, \|$, requiring (5.4) is equivalent to

$$\lim_{n\to\infty} \log\left(\sqrt{\sum_{i=1}^{K} (\lambda_i^n)^2} \left(\frac{1}{a^n}\right)^{\alpha}\right) = -\infty, \tag{5.5}$$

where λ_i denotes the eigenvalues of A. From (5.5) a straightforward calculation gives

$$\alpha \leq \frac{\log|\lambda|_{\max}}{\log a} = \frac{h_{\bar{F}}}{\mu_{\bar{F}}},$$

as claimed.

To prove the estimate on the capacity, we first note that by definition

$$C(\Lambda) = \liminf_{\varepsilon \to 0} \frac{\log n(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where $n(\varepsilon)$ is the number of intervals (1-balls) of length ε which is necessary to cover Λ. Using the covering constructed in the first part of the proof and letting $\epsilon = \delta(n)$, we can write

$$C(\Lambda) \leq \lim_{n\to\infty} \frac{\log\|A^n\|_s}{\log a^n} = \frac{1}{\log a} \lim_{n\to\infty} \frac{1}{n} \log\|A^n\|_s = \frac{\log|\lambda|_{\max}}{\log a},$$

where this last inequality follows from the spectral radius formula (see, e.g., [1]). □

In Theorem 5.2 we have used the topological definitions of the Haussdorff-dimension and the capacity. For cases when typical trajectories may not approach the invariant set in question, one can similarly define the metric versions of the entropy, Haussdorff-dimension, and the capacity (see, e.g., [9]). A notable fact is that for subshifts of finite type the topological entropy is the maximum of the metric entropies, as shown, e.g., in Mané [12] (in general, it is only the supremum of the metric entropies). If one can guarantee the existence of an invariant ergodic measure for the map F, then the (metric) Haussdorff-dimension of the invariant set exactly equals the quotient of the (metric) entropy and the Ljapunov exponent (see, e.g., [10] for a related result on one-dimensional, piecewise continuous maps). In view of this, we make the following conjecture:

Conjecture 5.1. *The estimates in the statement of Theorem 5.2 are in fact equalities, i.e.,*

$$HD(\Lambda) = C(\Lambda) = \frac{h_{\bar{F}}}{\mu_{\bar{F}}}.$$

In accordance with this, we have the estimate

$$h_{\bar{F}} < \log a. \tag{5.6}$$

We remark that estimate (5.6) holds in all the examples we considered with given values of the parameters a and b.

5.1. An Example

Let

$$a = 2.5, \qquad b = 2. \tag{5.7}$$

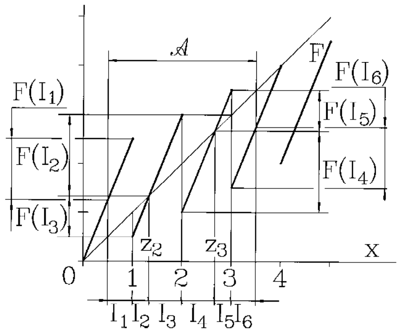

Fig. 10. μ-chaos map with $a = 5/2$, $b = 2$.

For these parameter values the graph of F is shown in Figure 10.

The formulas (2.6) and (2.5) provide $N = 3$ fixed points located at

$$z_1 = 0, \qquad z_2 = \frac{4}{3}, \qquad z_3 = \frac{8}{3}.$$

The formulas (3.4) and (3.5) in Proposition 3.2 with the definition (3.8) of the parameter domains of "instability" give

$$j_1 = j_2 = 2 \quad \Rightarrow \quad (a, b) = (2.5, 2) \in P_2^2.$$

Since

$$b = 2 \in \left[\frac{75}{38}, \frac{225}{112}\right] = (d_2^-(2.5), d_1^-(2.5)] \cap [d_3^+(2.5), d_2^+(2.5)) \cap [g_2^-(2.5), g_2^+(2.5)]$$

is satisfied in (3.24) under conditions (3.20) and (3.19), we have the parameters

$$(a, b) = (2.5, 2) \in Q_2^2,$$

satisfying the conditions of Theorem 3.8. Thus, the set

$$\mathcal{A} = \left[\frac{1}{2}, \frac{7}{2}\right]$$

defined in (3.9) is a hyperbolic strange attractor of F. Since F has no fixed points in $I - \mathcal{A} \cup \{0\}$, by (i) in Theorem 4.4 the domain of attraction for the attractor $\mathcal{A}$ is $I - \{0\}$, i.e., all trajectories starting away from the origin end up in the attractor.

The evaluation of formula (3.16) results in $k_0 = 0$, so the symbolic dynamics on $\mathcal{A}$ can be constructed with $K = 6$ intervals, as shown by (3.17). Based on the original partition (3.18), we obtain the Markov-partition (4.1) in the form

$$I_1 = \left[\frac{1}{2}, 1\right], \qquad I_2 = \left[1, \frac{4}{3}\right], \qquad I_3 = \left[\frac{4}{3}, 2\right],$$
$$I_4 = \left[2, \frac{8}{3}\right], \qquad I_5 = \left[\frac{8}{3}, 3\right], \qquad I_6 = \left[3, \frac{7}{2} - \varepsilon\right],$$

where $\varepsilon > 0$ small exists due to

$$b = 2 \in \left[\frac{75}{38}, \frac{225}{112}\right) \quad \Rightarrow \quad (a, b) = (2.5, 2) \in \hat{Q}_2^2 \subset Q_2^2.$$

Based on Proposition 4.1, the transition matrix takes the form

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

as can also be checked in Figure 10. This transition matrix satisfies the conditions of Proposition 5.1, which yields the preliminary estimate

$$h_{\bar{F}} \leq \log 3$$

for the topological entropy of F on Λ. However, Conjecture 4.3 suggests the refinement

$$h_{\bar{F}} \leq \log 2.5,$$

which agrees well with the numerical result

$$h_{\bar{F}} = \log |\lambda|_{\max} \simeq \log 2.32.$$

Using Theorem 5.2, the Haussdorff- and fractal dimensions of the hyperbolic set $\Lambda \subset \mathcal{A}$ obey the estimate

$$HD(\Lambda), C(\lambda) < \frac{\log 2.33}{\log 2.5}.$$

Again, Conjecture 5.3 suggests that we in fact have

$$HD(\Lambda) = C(\lambda) \simeq \frac{\log 2.32}{\log 2.5} = 0.92.$$

6. Conclusions

In this paper we studied the micro-chaos or μ-chaos map F defined by

$$x \mapsto ax - b\,\mathrm{Int}(x), \qquad x \in I = [0, m], \qquad 0 < a - 1 < b < a.$$

This map has a central role in describing the local dynamics of digitally controlled unstable systems. Such systems are subject to two discrete effects: sampling (a linear effect), and round-off errors (a nonlinear effect). These two effects frequently cause chaotic oscillations on a microscopic scale near a desired equilibrium of the system.

We proved the existence of a hyperbolic strange attractor for a large set of parameter values for the map F. We also studied its domain of attraction, which is of full measure, but may have a fractal boundary. This is due to the fact that the points not contained in the domain of attraction form invariant Cantor sets for certain parameter values (see Remark 4.1). We also described the dynamics on the attractor using symbolic dynamics. We identified regions in the parameter space with the same type of symbolic dynamics. This enabled us to draw an "instability chart" on the parameter plane to describe how the nature of chaotic dynamics changes as the parameters are varied (see Figures 6 and 9).

Whenever quantized-state control is used, the unstable equilibrium of the open loop remains unstable in the closed control loop. This instability does not necessarily result in chaos: Chaos can be suppressed, e.g., by dry-friction effects. These effects, however, may cause relatively large static errors in positioning. Hence one may have to put up with the presence of micro-chaos if one wants to stay in the regime of viable design parameters. In that case, our results can be used to reduce the size of the chaotic attractor, as well as its distance from the unstable equilibrium. This means reducing the amplitude and the mean value of chaotic oscillations to a level which is acceptable in a given problem. The detailed knowledge of the symbolic dynamics within the chaotic attractor makes it easier to identify statistical features of the irregular, micro-scale oscillations. This should also be of use in the design of more advanced control strategies.

It is to be noted that in numerical experiments with the map (2.1), the finite number of digits used in the computations are likely to introduce a further level of discretization which is not present in our original problem. As a result, simulations of the map may yield observable (i.e., stable) periodic solutions within the attractor $\mathcal{A}$, which of course contradicts the fact that the attractor is indecomposable. Related results can be found, e.g., in Domokos [7].

We finally comment on some related results of Delchamps [6]. He considered digitally controlled n-dimensional discrete problems and analyzed the one-dimensional case in more detail. He studied essentially the same map as our F_m in (1.5), but defined on both sides of the origin (this makes no difference since the map is odd). After identifying an attracting set (which contains our attractor $\mathcal{A}$), he proved the existence of an invariant ergodic measure for a *measure zero, nowhere-dense set of the parameter space* (a, b). The construction of the measure would be necessary to compute the related entropy and Haussdorff-dimension, but it is an unsolved problem in general. We note that Boyarsky and Scarowsky [4] construct invariant ergodic measures for certain Markov maps, but the micro-chaos map F is not Markov in their sense.

We believe that by proving the existence of a chaotic attractor for large sets of parameter values, constructing instability charts, and characterizing the strange attractor with its domain of attraction, our study lays the groundwork for the development of more advanced design principles for digitally controlled, one dimensional systems. An important direction for future research is the extension of the one-dimensional results to the higher dimensional digital control problems listed in the Introduction.

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